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Stacks and Real-Oriented Homotopy Theory

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ABSTRACT OF THE DISSERTATION

Stacks and Real-Oriented Homotopy Theory

by

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We begin by investigating analogues of the Ravenel conjectures in chromatic homotopy in the setting of Real-oriented homotopy theory, where one carries the data of canonical group actions by the cyclic group of order 2 via complex conjugation. This analysis yields a formula for Bousfield localization of C_2-spectra at the Real Johnson-Wilson theories, $E_{\mathbb{R}}(n)$, from which follows a smash product theorem and a chromatic convergence theorem for cofree C_2-spectra.

We turn to a systematic study of cofreeness in Real-oriented homotopy theory and establish the cofreeness of the norms of Real bordism theory, $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$, for all $n \geq 1$, recovering a result of Hu and Kriz at $n = 1$. The method of proof establishes a connection to the Segal conjecture for C_2 - also known as Lin's theorem - and yields a new, conceptual proof of this classical result.

We finish by bringing various equivariant spectra in Real-oriented homotopy theory into the world of stacks and chromatic homotopy by applying a method of Hopkins' to their fixed point spectra. We demonstrate this in detail for the Real Johnson-Wilson theories and give several modular descriptions of the stacks $\mathcal{M}_{ER(n)}$, recovering and generalizing Hopkins' description at $n = 1$ of $\mathcal{M}_{KO}$ as the moduli stack of nonsingular quadratic equations.

The dissertation of Christian Daniel Carrick is approved.

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To my brother and best friend, Dee

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This dissertation owes an enormous debt to my advisor, Mike Hill. This work depends on him intellectually in several ways, and his mathematical influence is evident throughout. In particular, it was his idea to establish the cofreeness of $MU^{((C_{2^n}))}$ using chromatic hypercubes, and he proved the $n = 1$ case in this way. Mike also observed the Bousfield equivalence

$$\langle E_{\mathbb{R}}(n)\rangle = \langle C_{2+} \wedge E(n)\rangle$$

and this was the starting point for all of the material in Chapter 4. Most of my understanding of equivariant homotopy theory I have learned more or less directly from Mike and his writings. Mike is a caring and honest advisor who has gone above and beyond for me since the very beginning of graduate school. He has deeply impacted the way I engage with mathematics on an intellectual, social, and political level.

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I would like to thank Paul Balmer for his excellent course on tensor-triangular geometry and for many helpful conversations related to the content of this thesis, especially that of Chapter 4. His perspective on categorical structures in homotopy theory has vastly influenced me.

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Chapter 1

INTRODUCTION

In this dissertation, we investigate the relationship between classical chromatic homotopy theory and its *equivariant* analogues in the context of Real-oriented homotopy theory. Real-oriented homotopy theory studies various complex-oriented cohomology theories together with their canonical complex conjugation actions, which lift these theories to genuine equivariant spectra known as Real-oriented cohomology theories. Chromatic homotopy is the study of complex-oriented cohomology theories, many of which are localizations of quotients of the complex bordism spectrum MU. Real-oriented homotopy theory thus began with Fujii [24] and Landweber [53] who defined the equivariant cohomology theory $MU_{\mathbb{R}}$, known as Real bordism theory, which lifts MU to a genuine C_2-spectrum acted on via complex conjugation. Several decades later, Hu-Kriz lifted many complex-oriented cohomology theories to the Real-oriented setting by constructing various localizations of quotients of $MU_{\mathbb{R}}$ - as C_2-spectra - and demonstrated their computational viability [48]. A notable example of a family of such Real-oriented theories is given by the Real Johnson-Wilson theories, $E_{\mathbb{R}}(n)$. Kitchloo-Wilson have studied these extensively and used them to give new results on non-immersions of real projective spaces [50].

In their landmark solution to the Kervaire Invariant problem, Hill-Hopkins-Ravenel (HHR) dramatically advanced this program by introducing larger groups of equivariance by use of symmetric monoidal structures [39]. In particular, they constructed a homotopically meaningful notion of tensor induction of G-spectra - known as the norm - and they used the norms of Real bordism theory $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$, a C_{2^n}-spectrum, to solve the Kervaire Invariant problem. They gave a powerful way of understanding these norms via the slice filtration, a filtration of a G-spectrum that lifts the Postnikov filtration of spectra and filters via representation spheres. They completely determined the slice filtration of the norms of Real bordism theory, allowing for a host of equivariant computations that were previously inaccessible.

Real-oriented homotopy and the slice filtration thus provide a strong analogy between chromatic homotopy in the classical setting and the corresponding approaches in the C_{2^n}-equivariant setting. We investigate the Ravenel conjectures in the context of this analogy and give an approach for understanding the *fixed points* of a Real-oriented cohomology theory in classical chromatic homotopy.

1.1 The Ravenel conjectures in Real-oriented homotopy theory

Chromatic homotopy has proven to be the dominant perspective on the large-scale, categorical structure of stable homotopy theory. This is most clearly demonstrated by the Ravenel conjectures, most of which were proven by Devinatz, Hopkins, and Smith [21]. Ravenel made a series of conjectures in [78] that were motivated by nilpotence and periodicity phenomena observed in the Adams-Novikov spectral sequence. We focus on two of these theorems: the smash product theorem and the chromatic convergence theorem. The smash product theorem states that for any spectrum X, the Bousfield localization functor at the n-th Johnson-Wilson theory $E(n)$ is given by the formula

$$L_{E(n)}(X) = L_{E(n)}(S^0) \wedge X$$

Localization at $E(n)$ is thus said to be *smashing*. This implies (and is equivalent to) the claim that $L_{E(n)}(-)$ commutes with colimits. The chromatic convergence theorem states that for a finite p-local spectrum X, the chromatic tower

$$\cdots \to L_{E(n)}(X) \to L_{E(n-1)}(X) \to \cdots \to L_{E(1)}(X) \to L_{E(0)}(X)$$

converges, in the sense that the canonical map

$$X \to \varprojlim L_{E(n)}(X)$$

is an equivalence. The latter theorem implies that the category of finite p-local spectra can be recovered from its chromatic localizations, and the former states that these localizations are well-behaved: in particular, they behave like Zariski localization at an open subscheme. In C_2-spectra, we establish the following analogous formulae.

Theorem 1.1.1. *Let $X \in \mathbf{Sp}^{C_2}$. The Bousfield localization of X at $E_{\mathbb{R}}(n)$ - the n-th Real Johnson-Wilson theory - is given by*

$$L_{E_{\mathbb{R}}(n)}(X) = F(EC_{2+}, L_{E_{\mathbb{R}}(n)}(S^0) \wedge X)$$

Theorem 1.1.2. *Let X be a finite 2-local C_2-spectrum. The $E_{\mathbb{R}}(n)$-chromatic tower at X converges to $F(EC_{2+}, X)$, in the sense that one has*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \varprojlim L_{E_{\mathbb{R}}(n)}(X) \\ F(EC_{2+}, X) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

We establish these formulae based on the observation that $E_{\mathbb{R}}(n)$ is actually Bousfield equivalent to the induced C_2-spectrum $C_{2+} \wedge E(n)$. We thus study the behavior of smashing Bousfield localizations along various change-of-group functors and obtain a necessary and sufficient condition for a smashing localization to remain smashing upon applying an induction functor, from which the above formulae follow.

These formulae are not an exact lift of the classical formulae, however. In each of them, it was necessary to apply the cofree functor $F(EC_{2+}, -)$ to get a correct formula. It may be more accurately stated that these are analogues of the Ravenel conjectures in *Borel* C_2-spectra, rather than genuine C_2-spectra.

1.2 Cofreeness in Real-oriented homotopy

$MU_{\mathbb{R}}$ and Real-oriented cohomology theories behave very similarly to their non-equivariant counterparts, as evidenced by the slice filtration. However, the above results indicate that the theorems we might expect to hold in Real-oriented homotopy hold only after applying the cofree functor. From another perspective, this is perhaps not unexpected, as Hu-Kriz showed that $MU_{\mathbb{R}}$ - the universal Real-oriented cohomology theory - is cofree [48]. We thus investigate the role of cofreeness in Real-oriented theories with larger groups of equivariance, as introduced by Hill-Hopkins-Ravenel [39]. We establish that the norms of Real bordism theory, $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ are cofree for all $n \geq 1$.

Theorem 1.2.1. *For all $n \geq 1$, the map of C_{2^n}-spectra*

$$N_{C_2}^{C_{2^n}} MU_{\mathbb{R}} \to F(EC_{2^n +}, N_{C_2}^{C_{2^n}} MU_{\mathbb{R}})$$

is an equivalence. That is, the C_{2^n}-spectrum $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ is cofree.

Our method is a proof of concept for applying chromatic techniques in the C_{2^n}-equivariant context. In particular, in chromatic homotopy, one may take chromatic localizations by inverting various v_n-classes on MU-module spectra. In Real-oriented homotopy, one has analogous classes $N(\overline{t_i})$ which one may invert on $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$-module spectra. It is easy to show that the localized theories $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}[N(\overline{t_i})^{-1}]$ are cofree, and the category of cofree C_{2^n}-spectra is closed under homotopy limits. Therefore, proceeding via a local cohomology approach, one has formally that the C_{2^n}-spectra

$$\tilde{L}_k(N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}) \coloneqq \mathrm{holim}_{\{i_1,\ldots,i_j\}\in\mathcal{P}_0([k])} N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}[(N(\overline{t_{i_1}} \cdots \overline{t_{i_j}})^{-1}]$$

formed via limits of local cohomology hypercubes are cofree, where $\mathcal{P}_0([k])$ is the poset of non-empty subsets of $[k] = \{1, \ldots, k\}$. By use of the slice theorem of Hill-Hopkins-Ravenel, we show that the map

$$N_{C_2}^{C_{2^n}} MU_{\mathbb{R}} \to \varprojlim \tilde{L}_k(N_{C_2}^{C_{2^n}} MU_{\mathbb{R}})$$

is an equivalence, establishing the result.

The Segal conjecture for C_2 was proven by Lin via difficult computations in the Adams spectral sequence [56]. Lin's theorem has received new attention recently as equivariant homotopy has experienced a renaissance following the Hill-Hopkins-Ravenel solution of the Kervaire Invariant problem (see [74], [67], and [35] for example). In particular, Nikolaus and Scholze established an equivalent formulation of Lin's theorem, showing that the most general form of the Segal conjecture for C_2 is equivalent to the claim that the C_2-spectrum $N_e^{C_2}H\mathbb{F}_2$ is cofree [74]. We use the equivalence of C_2-spectra $\Phi^{C_2}N_{C_2}^{C_4}BP_{\mathbb{R}} \simeq N_e^{C_2}H\mathbb{F}_2$ to show that Lin's theorem follows from the claim that $N_{C_2}^{C_4}MU_{\mathbb{R}}$ is cofree. This gives a new proof of Lin's theorem that is strongly conceptual and involves no homological algebra.

1.3 Stacks and chromatic measure

Real-oriented homotopy theory is built to mirror chromatic homotopy in the equivariant setting by observing that many of the known results in chromatic homotopy remain true when carrying a group action. In fact, the information flows the other way as well, in that non-equivariant homotopy is informed by the study of *fixed point* spectra of various Real-oriented theories. The phenomenon is perhaps best illustrated through the example KO, real topological K-theory. KO, unlike complex K-theory KU, is *not* complex-oriented, and therefore does not fit into the chromatic viewpoint as easily. Hopkins circumvents this by the following procedure: if E is a homotopy commutative ring spectrum with the property that the graded ring MU_*E is concentrated in even degrees, one may form the Hopf algebroid

$$(MU_*E, MU_*(MU \wedge E))$$

and call the associated stack $\mathcal{M}_E$ [22]. $\mathcal{M}_E$ comes equipped with a canonical affine morphism $\phi_E : \mathcal{M}_E \to \mathcal{M}_{FG}$ to the moduli stack of formal groups. This therefore associates to E a moduli problem related to formal groups. In the case KO, Hopkins establishes the surprising result that $\mathcal{M}_{KO}$ is equivalent to the moduli stack of nonsingular quadratic equations (see Proposition 6.3.4 in the case $n = 1$).

Our analysis begins with the observation that KO is in fact the fixed point spectrum of the Real-oriented cohomology theory $E_{\mathbb{R}}(1)$ (equivalently, Atiyah's Real K-theory $K_{\mathbb{R}}$), i.e.

$KO = E_{\mathbb{R}}(1)^{C_2}$. We thus have an obvious recipe to bring equivariant cohomology theories into the context of chromatic homotopy and moduli of formal groups: for $E \in \mathbf{Sp}^G$, we associate to E the stack $\mathcal{M}_{E^G}$. We study this stack for $E = E_{\mathbb{R}}(n)$ at all n, recovering Hopkins' result and relating larger n to various moduli problems.

Along the way, we define *chromatic measure*:

Definition 1.3.1. Let $X(n)$ be Ravenel's Thom spectrum over $\Omega SU(n)$ as in [79, 9.1]. For E a homotopy commutative ring spectrum, define the *chromatic measure* of E to be the integer

$$\Phi(E) = \min\{n \geq 0 : R \wedge X(n) \text{ is complex-oriented}\}$$

This is roughly a measure of how far E is from being complex-oriented, and it is straightforward to compute given knowledge of $\mathcal{M}_E$. We compute $\Phi(E)$ when $E = (E_{\mathbb{R}}(n))^{C_2}$ is the fixed points of the n-th Real Johnson-Wilson theory.

Theorem 1.3.2. $\Phi(E_{\mathbb{R}}(n)^{C_2}) = 2^n$. *In particular,* $E_{\mathbb{R}}(n)^{C_2} \wedge X(2^n)$ *is complex orientable.*

We also discuss an approach to computing $\Phi(BP_{\mathbb{R}}\langle n\rangle^{C_2})$ which will be the subject of future work. We demonstrate how to compute $\Phi(EO_n)$ for EO_n a higher real K-theory spectrum in terms of the valuation ν on the endomorphism ring $\mathrm{End}(\mathbb{G})$ of the corresponding formal group $\mathbb{G}$ of height n, and make an explicit calculation in a family of special cases corresponding to roots of unity. Specifically, if $n = k(p-1)$, one has a tower of division algebras

$$\mathbb{Q}_p \subset \mathbb{Q}_p(\zeta_p) \subset \mathrm{End}(\Gamma)[1/p]$$

Moreover, $\zeta_p^k - 1$ is a uniformizer of $\mathcal{O}_{\mathbb{Q}_p(\zeta_p)}$ and it follows that $\zeta_p^k - 1$ generates a C_p-subgroup of the Morava stabilizer group (see [86]). This C_p-subgroup acts on the corresponding Morava E-theory $E_{k(p-1)}$, and we have:

Theorem 1.3.3. *Let* $EO_{k(p-1)}$ *denote the homotopy fixed point spectrum* $(E_{k(p-1)})^{hC_p}$*, then* $\Phi(EO_{k(p-1)}) = p^k$. *In particular* $EO_{k(p-1)} \wedge X(p^k)$ *is complex orientable.*

We remark that the above approach is a natural way to bring equivariant ring spectra into the chromatic picture. In particular, if E is a G-spectrum, and $i_e^* E$ - the underlying spectrum of E - is complex-orientable, then there is a G-action on the formal group $\mathbb{G}_E$ corresponding to $i_e^* E$. The stacks approach records this automorphism data as a part of the geometry and allows us to compare the moduli problems $\mathrm{Spec}(\pi_{2*}(i_e^* E))/G$ and $\mathcal{M}_{E^G}$.

1.4 Organization of the dissertation

In the second and third chapters, we discuss preliminaries needed for our results on genuine G-spectra. Chapter 2 begins with a detailed exposition of the theory of stacks, with an emphasis on locally presentable stacks and their relationship to Hopf algebroids. We give a detailed analysis of the geometry of the moduli stack of formal groups $\mathcal{M}_{FG}$ and detail the chromatic approach to stable homotopy theory from the viewpoint of stacks and the Ravenel conjectures. Chapter 3 sets up the foundations for genuine equivariant stable homotopy theory with an emphasis on $G = C_{2^n}$. In particular, we review the theory of Real orientations of C_2-spectra and discuss analogous notions developed for C_{2^n}-spectra in [39], from the point of view of the slice filtration. The chapter finishes by giving a proof that the classical Segal conjecture for C_p is implied by the claim that the Tate diagonal

$$S^0 \to (S^0)^{tC_p}$$

is a p-complete equivalence. In Chapter 4, we establish the analogues of the smash product and chromatic convergence theorems in cofree C_2-spectra by a general analysis of smashing localizations and their behavior under change-of-group functors. This chapter follows closely the author's results in [18]. In Chapter 5, we establish the cofreeness of the norms of Real bordism theory, using the slice theorem of Hill-Hopkins-Ravenel, and we show that this gives a new proof of the Segal conjecture for C_2. This chapter follows closely the author's results in [17]. We finish in Chapter 6 with a detailed analysis of the above stacks approach in the case when $E = E_{\mathbb{R}}(n)$ is the n-th Johnson-Wilson theory. We also introduce chromatic measure and compute its value on the fixed points of the $E_{\mathbb{R}}(n)$'s and on higher real K-theory EO_n spectra.

Chapter 2

STACKS AND CHROMATIC HOMOTOPY

In this chapter, we introduce the basics of the chromatic approach to stable homotopy, from the point of view of stacks. There is a wealth of good exposition on this material: we refer the reader to [58] and [44], and we follow both of these closely. In Section 2.1, we begin squarely in the context of algebraic geometry by introducing stacks and studying various properties through the example of $\mathcal{M}_{FG}$, the moduli stack of formal groups, the key example in chromatic homotopy. In Section 2.2, we turn to topology and review the basics of chromatic homotopy theory and its connection to the theory of stacks. With this in place, in Section 2.3, we state the Ravenel conjectures and explain how the theory of stacks gives a powerful viewpoint on these results.

2.1 Stacks

Stacks are a ubiquitous tool in algebraic geometry as they give an effective way to study moduli problems. In particular, the theory of stacks admits much of the same flexibility as the theory of *schemes*, while carrying significantly more data relevant to various moduli problems. Stacks begin from the basic observation that to understand a moduli problem properly, one must remember not only the isomorphism classes of a particular algebro-geometric object, but also the nontrivial automorphisms carried by such an object (see [71] for a discussion of this in the case of the moduli of curves). Indeed, this is necessary to have suitable forms of *descent* available for moduli problems and, in particular, to have a category in which suitable moduli spaces exist.

In Section 2.1.1 we review and motivate the definition of descent, stacks, and stackification, and we give a number of examples. In Section 2.1.2, we turn to our main class of examples of stacks: locally presentable stacks. These are essentially the same data as a Hopf algebroid, and we make this connection precise and connect quasicoherent sheaves over a locally presentable stack to comodules over the corresponding Hopf algebroid, in Section 2.1.3. We turn to the example of interest in chromatic homotopy, $\mathcal{M}_{FG}$ in Section 2.1.4, and we show in Section 2.1.5 that, after localizing at a prime, $(\mathcal{M}_{FG})_{(p)}$ admits a filtration by height which strongly controls the geometry of the stack.

To motivate stacks more precisely, we begin with the following informal definition. In this section, we fix a commutative ring k and let $\mathbf{Aff} := \mathbf{CAlg}_k^{op}$ be the category of affine schemes over k. For a commutative k-algebra R, we let $\mathrm{Spec}(R)$ be the object in $\mathbf{Aff}$ corresponding to R.

Definition 2.1.1. (Informal) A *moduli problem* is a functor $\mathcal{F} : \mathbf{Aff}^{op} \to \mathbf{Sets}$. A solution to the moduli problem $\mathcal{F}$ is a categorical framework in which

- The functor $\mathcal{F}$ is a representable.
- The values of $\mathcal{F}$ are determined by local data.

We make sense of this definition through several examples.

Example 2.1.2. The functor $\mathbb{G}_m : \mathbf{Aff}^{op} \to \mathbf{Sets}$ given by $\mathrm{Spec}(R) \mapsto R^\times$ is represented by $\mathrm{Spec}(k[t, t^{-1}])$; we did not need to alter the categorical framework to solve this moduli problem. The locality condition is satisfied because $\mathbb{G}_m$ is a sheaf, which is to say that if $\{U_i \to \mathrm{Spec}(R)\}$ is any open cover, one has an equalizer sequence of sets

$$\mathbb{G}_m(\mathrm{Spec}(R)) \to \prod_i \mathbb{G}_m(U_i) \implies \prod_{i,j} \mathbb{G}_m(U_i \times_{\mathrm{Spec}(R)} U_j)$$

Example 2.1.3. The functor $\mathbb{P}^n : \mathbf{Aff}^{op} \to \mathbf{Sets}$ sending

$$\begin{aligned}\mathrm{Spec}(R) &\mapsto \{\text{“Lines through the origin” in } R^{n+1}\} \\ &:= \{\text{Rank one projective } R\text{-module quotients of } R^{n+1}\}\end{aligned}$$

is not represented by an affine scheme, but if we enlarge $\mathbf{Aff}^{op}$ to $\mathbf{Sch}$, the category of schemes over k, it is represented by the scheme given the same name: $\mathbb{P}^n$, which is again a sheaf.

Example 2.1.4. The functor $\mathrm{LB} : \mathbf{Aff}^{op} \to \mathbf{Sets}$ given by

$$\mathrm{Spec}(R) \mapsto \{\text{Isomorphism classes of line bundles on } \mathrm{Spec}(R)\}$$

cannot be represented by an object in the category of schemes over k. If it were, $\mathrm{LB}(-)$ would be a sheaf as in the previous examples, and hence for any open cover $\{U_i \to \mathrm{Spec}(R)\}$, we would have an equalizer sequence of sets

$$\mathrm{LB}(\mathrm{Spec}(R)) \xrightarrow{\iota} \prod_i \mathrm{LB}(U_i) \to \prod_{i,j} \mathrm{LB}(U_i \times_{\mathrm{Spec}(R)} U_j)$$

In particular, the map ι in the above sequence would be an injection. However, if $\mathcal{L}$ is a line bundle on $\mathrm{Spec}(R)$, we may choose an open cover $\{U_i\}$ such that $\mathcal{L}|_{U_i}$ is trivial for all i, and hence the tuple

$$(\mathcal{L}|_{U_i}) \in \prod_i \mathrm{LB}(U_i)$$

is *equal* to the tuple (triv_i) where triv_i is the trivial line bundle over U_i. But we have

$$\iota(\mathcal{L}) = (\mathcal{L}|_{U_i}) = (\mathrm{triv}_i) = \iota(\mathrm{triv})$$

where triv is the trivial line bundle over $\mathrm{Spec}(R)$. Since ι is an injection, this would imply that every line bundle $\mathcal{L}$ is trivial.

However, we can recover a line bundle from the data of its local trivializations, hence if we ask that our categorical framework remember the isomorphisms

$$\mathcal{L}|_{U_i} \cong U_i \times \mathrm{Spec}(R)$$

for a line bundle $\mathcal{L}$ and an open cover $\{U_i \to \mathrm{Spec}(R)\}$, we may resolve this issue.

We therefore instead enlarge **Sets** and replace LB with the functor

$$\widetilde{\mathrm{LB}} : \mathbf{Aff}^{op} \to \mathbf{Groupoids}$$

that sends $\mathrm{Spec}(R)$ to the groupoid of line bundles on $\mathrm{Spec}(R)$. Recording the groupoid of line bundles - instead of just the isomorphism classes - allows one to remember the local trivializations of a line bundle. As we will see, if we enlarge $\mathbf{Aff}^{op}$ to the category of stacks on $\mathbf{Aff}^{op}$, then there is an equivalence of groupoids

$$\mathrm{Hom}_{\mathbf{Stacks}}(\mathrm{Spec}(R), B\mathbb{G}_m) \simeq \widetilde{\mathrm{LB}}(\mathrm{Spec}(R))$$

and in particular a bijection

$$\pi_0(\mathrm{Hom}_{\mathbf{Stacks}}(\mathrm{Spec}(R), B\mathbb{G}_m)) \cong \mathrm{LB}(\mathrm{Spec}(R))$$

where π_0 denotes the set of isomorphism classes of a groupoid.

2.1.1 Prestacks, stacks, and stackification

There are several ways to approach the theory of stacks, and in this section we take the approach that we feel is the most down-to-earth from the perspective of chromatic homotopy. In particular, we use the functor-of-points approach. It is common to use, alternatively, the theory of fibered categories (see [85]); this is a more geometric approach which may be preferable to algebraic geometers and for which there is a more robust literature. However, the approaches are equivalent via the Grothendieck construction: see Remark 2.1.20 below.

We also use *strict* functors to **Groupoids** as opposed to pseudofunctors. Algebraic geometers will also find this somewhat nonstandard, and it is admittedly a less flexible approach in general. For instance $\widetilde{\mathrm{LB}}(-)$ from Example 2.1.4 is not a strict functor, though we discuss a strictification of it below. However, this is sufficient and convenient for our approach, and the two approaches produce equivalent theories, as stacks may always be strictified (see again Remark 2.1.20 below).

As advertised, stacks are meant to provide solutions to certain moduli problems while behaving formally similarly to schemes. According to Definition 2.1.1, we must first make process the notion of *locality*, which leads us to the following definition.

Definition 2.1.5. Let $\mathcal{C}$ be a category with finite limits. A *Grothendieck topology* on $\mathcal{C}$ is a collection J of sets of morphisms $\{U_i \to U\}$ in $\mathcal{C}$ called "coverings" satisfying the following properties:

1. For any isomorphism $f : U \to V$ in $\mathcal{C}$, the set $\{f : U \to V\}$ is a covering.

2. Transitivity: if $\{U_i \to U\}$ is a covering, and for each i, $\{V_{ij} \to U_i\}$ is a covering, then the set of composites $\{V_{ij} \to U\}$ is a covering.

3. Closure under pullbacks: if $\{U_i \to U\}$ is a covering, and $V \to U$ is any morphism, then $\{V \times_U U_i \to V\}$ is a covering.

We will refer to a pair $(\mathcal{C}, J)$ as a *Grothendieck site*, or just a *site*.

Example 2.1.6. The Zariski topology on $\mathcal{C} = \mathbf{Aff}$ is a Grothendieck topology, where a set of morphisms

$$\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}$$

is a covering if each $\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$ is a Zariski open immersion, and the map

$$\coprod_i \mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$$

is surjective.

Remark 2.1.7. In the above definition, we use sets of morphisms $\{U_i \to U\}$ as this matches our intuition about open covers from topology. In practice, however, it is often more convenient to speak of a single morphism $V \to U$ as a cover, where one replaces $\{U_i \to U\}$ with the map

$$V = \coprod U_i \to U$$

This requires $\mathcal{C}$ to have (arbitrary) coproducts, but this will always be the case in our applications, as $\mathbf{Aff}$ has arbitrary coproducts. Working in this way, one may equivalently define a topology J on $\mathcal{C}$ to be a subcategory $J \subset \mathcal{C}$ that is wide, replete, and stable under pullbacks. In Example 2.1.6, this has the effect of replacing the set of morphisms $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}$ with the single morphism

$$\mathrm{Spec}(\prod A_i) \cong \coprod \mathrm{Spec}(A_i) \to \mathrm{Spec}(A)$$

We make use of both approaches.

Example 2.1.8. We will primarily work with the faithfully flat topology (or just the flat topology, for short) on $\mathcal{C} = \mathbf{Aff}$. Here a cover is a map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ that is dual to a faithfully flat map of k-algebras $A \to B$. Equivalently, $A \to B$ is flat and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

If $\mathcal{C}$ is equipped with a Grothendieck topology $(\mathcal{C}, J)$, one has a notion of *locality* in $\mathcal{C}$, and this determines a corresponding notion of *descent* in $\mathcal{C}$. Informally, we say a functor $\mathcal{F}$ on $\mathcal{C}$ satisfies descent with respect to J if the value of $\mathcal{F}$ on $U \in \mathcal{C}$ may be recovered - uniquely in some sense - from the values $\mathcal{F}(U_i)$ for a cover $\{U_i \to U\}$. Making this descent condition precise depends on what sort of functor $\mathcal{F}$ is. The most straightforward example is that of a sheaf (of sets).

Definition 2.1.9. Let $\mathcal{F} : \mathcal{C}^{op} \to \mathbf{Sets}$ be a presheaf on $\mathcal{C}$. We say $\mathcal{F}$ is a *sheaf* on $(\mathcal{C}, J)$, if for every cover $p : V \to U$, the following diagram is an equalizer in **Sets**:

$$\mathcal{F}(U) \xrightarrow{p^*} \mathcal{F}(V) \overset{\pi_1^*}{\underset{\pi_2^*}{\rightrightarrows}} \mathcal{F}(V \times_U V)$$

where $\pi_i : V \times_U V \to V$ for $i = 1, 2$ are the canonical projections.

For the reader familiar with sheaves in the classical setting of affine schemes - sheaves on $(\mathbf{Aff}, \mathbf{Zariski})$ - perhaps the first question that comes to mind is how sheafification works in this more general context. We will need the following definition.

Definition 2.1.10. For $U \in \mathcal{C}$, let $\mathbf{Cover}(U)$ be the category of covers $\{p : V \to U\} \in J$, where a morphism $\{p : V \to U\} \to \{p' : V' \to U\}$ is a commutative diagram

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[d] & U \\ V' \arrow[ur, "p'"'] & \end{tikzcd}$$

Definition 2.1.11. (Grothendieck's plus construction) Let $(\mathcal{C}, J)$ be a site and $\mathcal{F}$ a presheaf on $\mathcal{C}$. For $\{p : V \to U\} \in J$, define

$$H^0(p, \mathcal{F}) := \{s \in \mathcal{F}(V) \, : \, \pi_1^*(s) = \pi_2^*(s) \in \mathcal{F}(V \times_U V)\}$$

Let $\mathcal{F}^+$ be the presheaf on $\mathcal{C}$ defined by

$$\mathcal{F}^+(U) = \mathrm{colim}_{p \in \mathbf{Cover}(U)^{op}} H^0(p, \mathcal{F})$$

Remark 2.1.12. $\mathcal{F}^+$ is not quite a sheaf on $(\mathcal{C}, J)$. It is, however, a *separated* presheaf: for a covering $\{p : V \to U\} \in J$, the map $p^* : \mathcal{F}^+(U) \to \mathcal{F}^+(V)$ is an injection. Moreover, if $\mathcal{F}$ is already separated, then $\mathcal{F}^+$ is a sheaf (see [85, tag 00W1] for more details). Therefore $\mathcal{F}^{++}$ is the sheafification of $\mathcal{F}$, and the functor $(-)^{++}$ exhibits the category of sheaves on $(\mathcal{C}, J)$ as a reflective subcategory of $\mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Sets})$, the category of presheaves on $\mathcal{C}$.

A stack is, suitably defined, a sheaf of *groupoids* instead of sets, and one may keep Example 2.1.4 in mind to parse and motivate the following definitions. We begin first with the analogous notion of presheaf in this setting.

Definition 2.1.13. A *prestack* on $\mathcal{C}$ is a (strict) functor $\mathcal{C}^{op} \to \mathbf{Groupoids}$. Here $\mathbf{Groupoids}$ is the category of Groupoids and (strict) natural transformations.

Remark 2.1.14. According to the above definition, a prestack is merely a presheaf of groupoids. We can be more specific about what we mean by the category of prestacks on $\mathcal{C}$, however, as the obvious such category has a natural refinement to a strict 2-category. For F, G prestacks on $\mathcal{C}$, there is a groupoid $\mathrm{Hom}_{\mathbf{Prestacks}}(F, G)$ as follows.

- The objects of $\mathrm{Hom}_{\mathbf{Prestacks}}(F, G)$ are (strict) natural transformations of functors $T : F \implies G$, hence a natural family of functors $T_x : F(x) \to G(x)$ for $x \in \mathcal{C}$.

- A morphism in $\mathrm{Hom}_{\mathbf{Prestacks}}(F,G)$ (i.e. a 2-morphism in **Prestacks**) from T to S for $T,S : F \to G$ consists of a natural transformation $\alpha_x : T_x \to S_x$ for every $x \in \mathcal{C}$, satisfying the following naturality property: suppose $f : x \to y$ is a morphism in $\mathcal{C}$, then we ask for the following natural transformations of functors $F(x) \to G(y)$ to be equal

$$\begin{tikzcd} F(x) \arrow[r, bend left, "S_x"] \arrow[r, bend right, "T_x"'] \arrow[r, phantom, "\Downarrow\alpha_x"] & G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd}$$

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] & F(y) \arrow[r, bend left, "S_y"] \arrow[r, bend right, "T_y"'] \arrow[r, phantom, "\Downarrow\alpha_y"] & G(y) \end{tikzcd}$$

To define a stack, we want an appropriate notion of descent in **Prestacks** relative to a given topology J on $\mathcal{C}$. A naive guess would be to ask, for a prestack $\mathcal{F}$ on $\mathcal{C}$ and each cover $\{p : V \to U\} \in J$, for the sequence

$$\mathcal{F}(U) \to \mathcal{F}(V) \implies \mathcal{F}(V \times_U V)$$

to be an equalizer in **Groupoids**. Since we have chosen to work entirely in the strict setting, this turns out to be a necessary condition for $\mathcal{F}$ to be a stack, although it is not sufficient and is in any case far too rigid a notion of descent. Indeed, consider again Example 2.1.4: when $\mathcal{F} = \widetilde{\mathrm{LB}}$. Our notion of descent should capture the fact that a line bundle may be recovered, up to isomorphism, by cocycle data. The above condition would tell us that a line bundle may be recovered only by trivial cocycle data, in which the isomorphisms on the intersection $V \times_U V$ are identity maps.

This points to two features the above descent condition fails to capture. First, for a line bundle L on V, we should not ask for the restrictions $\pi_1^* L$ and $\pi_2^* L$ to be *equal* on the intersection $V \times_U V$; rather we should ask for an isomorphism

$$\alpha : \pi_1^* L \xrightarrow{\cong} \pi_2^* L$$

on the intersection $V \times_U V$. Second, the isomorphism α needs to satisfy the cocycle condition on the triple intersection $V \times_U V \times_U V$. The latter condition tells us we should be asking for a universal property on the larger diagram

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V) \Rrightarrow \mathcal{F}(V \times_U V \times_U V)$$

and the former condition tells us that we should not ask for this to be a limit diagram, in the strict 1-categorical sense. Our notion of descent needs to be defined with reference to the weaker notion of equivalence in **Groupoids** - that of an equivalence of categories - as opposed to the profane notion of an *isomorphism* of categories. This, of course, amounts to asking for the above diagram to be a sort of *homotopy* limit diagram. Conveniently, however, in **Groupoids**, one may define such notions directly without being precise about a homotopy theory of groupoids.

Definition 2.1.15. Let $(\mathcal{C}, J)$ be a site, $\{p : V \to U\} \in J$ a cover, and

$$\mathcal{F} : \mathcal{C}^{op} \to \textbf{Groupoids}$$

a prestack on $\mathcal{C}$. We define the *descent groupoid* $\mathrm{Desc}_p(\mathcal{F})$ as follows:

- An object of $\mathrm{Desc}_p(\mathcal{F})$ is an object $E \in \mathcal{F}(V)$ together with an isomorphism

  $$\alpha : \pi_1^* E \to \pi_2^* E$$

  in $\mathcal{F}(V \times_U V)$ satisfying the cocycle condition: $\alpha_{23} \circ \alpha_{12} = \alpha_{13}$ as morphisms in the groupoid

  $$\mathcal{F}(V \times_U V \times_U V)$$

  Here we let $\pi_{ij} : V \times_U V \times_U V \to V \times_U V$ be the projection onto the i and j-th factor for $i < j$, and $\alpha_{ij} := \pi_{ij}^*(\alpha)$.

- A morphism $\phi : (E, \alpha) \to (E', \alpha')$ consists of a morphism $\phi : E \to E'$ in $\mathcal{F}(V)$ such that the diagram

$$\begin{tikzcd} \pi_1^* E \arrow[r, "\pi_1^*(\phi)"] \arrow[d, "\alpha"] & \pi_1^* E' \arrow[d, "\alpha'"] \\ \pi_2^* E \arrow[r, "\pi_2^*(\phi)"] & \pi_2^* E' \end{tikzcd}$$

commutes.

For $\mathcal{F}$ a prestack on $\mathcal{C}$, functoriality of $\mathcal{F}$ defines restriction maps that assemble into a functor $\mathcal{F}(U) \to \mathrm{Desc}_p(\mathcal{F})$ for a covering $\{p : V \to U\} \in J$.

Definition 2.1.16. We say $\mathcal{F}$ is a *stack* if, for each covering in $\{p : V \to U\} \in J$, the functor

$$\mathcal{F}(U) \to \mathrm{Desc}_p(\mathcal{F})$$

is an equivalence of groupoids.

Proposition 2.1.17. *The strict 2-category of stacks is a full sub-2-category of* **Prestacks***, and there exists a "stackification" functor*

$$(-)^a : \mathbf{Prestacks} \to \mathbf{Stacks}$$

satisfying the following universal property: we have an equivalence of groupoids

$$\mathrm{Hom}_{\mathbf{Stacks}}(\mathcal{F}^a, \mathcal{G}) \simeq \mathrm{Hom}_{\mathbf{Prestacks}}(\mathcal{F}, \mathcal{G})$$

for $\mathcal{G}$ a stack, and these equivalences are natural in $\mathcal{G}$.

Proof. We summarize the construction given in [85, tag 02ZN]. Let

$$\mathcal{F} : \mathcal{C}^{op} \to \mathbf{Groupoids}$$

be a prestack. One begins by sheafifying this functor, which may be done by sheafifying the object and morphism presheaves (of sets). One then applies a stacky variant of the plus construction by setting, for $U \in \mathcal{C}$

$$(\mathcal{F})^a(U) = \mathrm{hocolim}_{p \in \mathbf{Cover}(U)^{op}} \mathrm{Desc}_p(\mathcal{F})$$

This requires, of course, an appropriate notion of homotopy colimit in **Groupoids**, but once again it is straightforward to be explicit in this case. This has the additional advantage of giving, for $\mathcal{F}$ a strict prestack, an explicit model of $(\mathcal{F})^a$ that is also strict. One sets

$$\mathrm{ob}(\mathcal{F})^a(U) = \{(\{p : V \to U\}, (E,\alpha)) \mid p \in J \text{ and } (E,\alpha) \in \mathrm{Desc}_p(\mathcal{F})\}$$

That is, an object of $(\mathcal{F})^a(U)$ is a choice of cover of U and cocycle datum for $\mathcal{F}$ with respect to this cover. For the set of morphisms

$$(\{p : V \to U\}, (E,\alpha)) \to (\{p' : V' \to U\}, (F,\beta))$$

we say a cover $\{q : W \to U\} \in J$ is a common refinement of p and p' if there is a diagram in **Cover**(U)

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"'] & V' \arrow[d, "p'"] \\ V \arrow[r, "p"'] & U \end{tikzcd}$$

such that $p \circ g = p' \circ f = q$. The set of such common refinements determines an obvious (co-filtered) category we call $\mathcal{U}$, and we define

$$\mathrm{Hom}_{(\mathcal{F})^a(U)}((p,(E,\alpha)),(p',(F,\beta)))$$

to be the set

$$\mathrm{colim}_{q \in \mathcal{U}^{op}} \mathrm{Hom}_{\mathrm{Desc}_q(\mathcal{F})}(f^*(E,\alpha), g^*(F,\beta))$$

□

Definition 2.1.18. A morphism of (pre)-stacks $f : \mathcal{M} \to \mathcal{N}$ is said to be an *equivalence* if it is pointwise. That is, for all $c \in \mathcal{C}$, $f(c) : \mathcal{M}(c) \to \mathcal{N}(c)$ is an equivalence of groupoids.

Definition 2.1.19. We say a groupoid is *discrete* if it has no non-identity automorphisms. We say a prestack $\mathcal{M}$ is discrete if the groupoid $\mathcal{M}(\mathrm{Spec}(R))$ is discrete for all nonzero k-algebras R. Note that if a stack $\mathcal{M}$ is discrete as a prestack, the groupoid $\mathcal{M}(\mathrm{Spec}(*))$ is necessarily discrete, where $*$ denotes the zero ring.

Remark 2.1.20. We are due now for a remark on our choice to work in the strict setting. Most of the stacks appearing in chromatic homotopy come to us as stacks associated to a groupoid object in **Schemes**, and all such stacks are strict. Moreover, most of the constructions we perform - e.g. homotopy pullbacks, homotopy inverse limits, and quotient stacks - admit straightforward explicit models, so this doesn't really limit our flexibility. Every prestack is canonically naturally equivalent to one that is a strict functor - i.e. it can always be strictified in a functorial way - see [85, tag 004A].

Remark 2.1.21. For the reader who prefers not to work in the strict setting and prefers a homotopy theoretic approach to stackification, see Hollander [43]. Hollander equips **Prestacks** with the structure of a model category in which **Stacks** is precisely the subcategory of fibrant objects, and stackification corresponds to fibrant replacement.

We finish the subsection with some basic examples of stacks. Unless otherwise stated, we'll be working over the site $(\mathbf{Aff}, \mathbf{Flat})$.

Example 2.1.22. *Representable stacks*: Every affine scheme $\mathrm{Spec}(S)$ determines a stack (which we still denote by $\mathrm{Spec}(S)$) via the representable functor $\mathrm{Hom}_{\mathbf{Aff}}(-, \mathrm{Spec}(S))$, where regard the set $\mathrm{Hom}_{\mathbf{Aff}}(\mathrm{Spec}(R), \mathrm{Spec}(S))$ as a groupoid with only identity morphisms. For any stack Y, by Yoneda, we have an equivalence of groupoids:

$$\mathrm{Hom}_{\mathbf{Stacks}}(\mathrm{Spec}(S), Y) \simeq Y(\mathrm{Spec}(S))$$

Example 2.1.23. *BG*: Let G be a finite group. Let BG^{pre} denote the prestack that is a constant functor at the groupoid consisting of a single object with automorphism group G. We let BG denote the stackification of BG^{pre}. Following our construction of stackification above, for X an affine scheme, an object in $BG(X)$ is a choice of faithfully flat cover $\{U \to X\}$ and a locally constant function $\alpha : U \times_X U \to G$, so that α satisfies the cocycle condition on $U \times_X U \times_X U$. This is a G-cocycle datum on X, which is equivalent to a principal G-bundle / G-torsor on X.

A map between two such cocycles consists of a common refinement, and a map of cocycles on the refinement. Such a map is the same data as a map of G-torsors on X. We find therefore that $BG(X)$ is equivalent to the groupoid of G-torsors on X, hence BG is the moduli stack of G-torsors. One can of course do this and the next example with more generally a group scheme - for instance with $\mathbb{G}_m$, giving a strict model of $\widetilde{\mathrm{LB}}$ from Example 2.1.4.

Example 2.1.24. *Quotient stacks*: Let G be a finite group acting on a scheme $X \in \mathbf{Aff}$. We will define a quotient stack X/G. Usually when we take quotients by group actions, on say a space, we glue together points that are in the same orbit. Here we are working with groupoids, so we don't want to glue objects together, we want to build in an isomorphism between them instead. This leads us to the following.

Suppose S is a G-set, let $B_G(S)$ denote the *action groupoid* of S, which has objects S and

$$\mathrm{Hom}_{B_G(S)}(s_1, s_2) = \{g \in G : g \cdot s_1 = s_2\}$$

Since G acts on the scheme X, the set $\mathrm{Hom}_{\mathbf{Aff}}(\mathrm{Spec}(R), X)$ inherits a G-action, for any $\mathrm{Spec}(R) \in \mathbf{Aff}$. We therefore define the prestack $(X/G)^{pre}$ by

$$(X/G)^{pre}(\mathrm{Spec}(R)) = B_G(\mathrm{Hom}_{\mathbf{Aff}}(\mathrm{Spec}(R), X))$$

and then (X/G) is the stackification of $(X/G)^{pre}$. It's not hard to see for example that

$$BG \simeq \mathrm{Spec}(k)/G$$

where $\mathrm{Spec}(k)$ is given the trivial G-action. For this reason, BG is sometimes referred to as $*/G$ since $\mathrm{Spec}(k)$ is the terminal object in $\mathbf{Aff}$.

2.1.2 Locally presentable stacks

In this section, we discuss some basic properties of stacks and use these to define the notion of a *locally presentable stack*. We show that any such stack admitting an affine cover is equivalent to a stack associated to a Hopf algebroid. We continue to work over the site $(\mathbf{Aff}, \mathbf{Flat})$ over a fixed commutative ring k, where $\mathbf{Aff} := \mathbf{CAlg}_k$.

We begin with the homotopy pullback of stacks. Whenever we say a square diagram of groupoids commutes, we mean that it commutes up to a (possibly unspecified) 2-isomorphism. This means that in a diagram of groupoids of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "g"] \\ C \arrow[r, "k"] & D \end{tikzcd}$$

there is a natural transformation of functors $T : g \circ f \implies k \circ h$ from $A \to D$. In practice, this 2-isomorphism T is often understood from context, such as when the above square is a homotopy pullback square of groupoids, which we now define.

Definition 2.1.25. Suppose given a diagram

$$\begin{tikzcd} & B \arrow[d, "F"] \\ C \arrow[r, "G"'] & D \end{tikzcd}$$

of groupoids, then the *homotopy pullback* $B \times_D C$ is the groupoid given as follows

- $\mathrm{ob}(B \times_D C)$ consists of triples (b, c, ϕ), where $b \in B, c \in C$, and ϕ is an isomorphism $F(b) \xrightarrow{\phi} G(c)$.

- A morphism of triples $(b, c, \phi) \to (b', c', \phi')$ consists of a morphism $f : b \to b'$ in B and a morphism $g : c \to c'$ in C such that the following diagram commutes

$$\begin{tikzcd} F(b) \arrow[r, "F(f)"] \arrow[d, "\phi"'] & F(b') \arrow[d, "\phi'"] \\ G(c) \arrow[r, "G(g)"] & G(c') \end{tikzcd}$$

Note that in the pullback square

$$\begin{tikzcd} B \times_D C \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

there is a *canonical* 2-isomorphism making the diagram commute: it is given on an object (b, c, ϕ) by the isomorphism ϕ.

Remark 2.1.26. When we say a commutative square of groupoids

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & B' \arrow[d] \\ C' \arrow[r] & D' \end{tikzcd}$$

is a pullback square, we mean that there is a morphism of commutative squares

$$\begin{tikzcd}[row sep=small, column sep=small] & B \times_D C \arrow[rr] \arrow[dd] & & B \arrow[dd] \\ \mathcal{P} \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & B' \arrow[ur] & \\ & C \arrow[rr] & & D \\ C' \arrow[ur] \arrow[rr] & & D' \arrow[ur] \arrow[from=uu, crossing over] & \end{tikzcd}$$

in which each square commutes up to a (possibly unspecified) 2-isomorphism, and each arrow from the front square to the back square is an equivalence of groupoids.

Definition 2.1.27. The *homotopy pullback of stacks* is then done pointwise on a diagram of stacks as above. That is, given a diagram

$$\begin{tikzcd} & \mathcal{N} \arrow[d] \\ \mathcal{N}' \arrow[r] & \mathcal{M} \end{tikzcd}$$

of (pre)-stacks, we define the homotopy pullback $\mathcal{N}' \times_{\mathcal{M}} \mathcal{N}$ to be the prestack given by

$$(\mathcal{N}' \times_{\mathcal{M}} \mathcal{N})(\mathrm{Spec}(R)) = \mathcal{N}'(\mathrm{Spec}(R)) \times_{\mathcal{M}(\mathrm{Spec}(R))} \mathcal{N}(\mathrm{Spec}(R))$$

with the obvious functoriality.

Remark 2.1.28. We will rarely speak of a sort of pullback of stacks other than this one, so we will often drop the prefixes “stacky” or “homotopy”. We will make use of the fact that pullback of prestacks commutes with stackification (see [85, tag 04Y1]). In particular, if $\mathcal{N}$, $\mathcal{N}'$, and $\mathcal{M}$ are stacks, then $\mathcal{N}' \times_{\mathcal{M}} \mathcal{N}$ is a stack.

Remark 2.1.29. It is an instructive exercise to check that this definition of homotopy pullback is indeed *homotopy invariant*. That is, if one replaces B, C, or D with an equivalent groupoid, the corresponding homotopy pullback remains equivalent to $B \times_D C$.

It is often useful to know that, in the category **Top** of topological spaces, a Serre fibration $E \to B$ has the property that any strict pullback square

$$\begin{tikzcd} A \times_B E \arrow[r] \arrow[d] & E \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

is also a *homotopy* pullback square. A similar fact is true in **Groupoids** and therefore in **Stacks**. We say a morphism of groupoids $F : B \to D$ is a fibration if, for all $b \in B$, and for all morphisms $f : F(b) \to d$ in D, there exists a morphism $\tilde{f} : b \to b'$ in B such that $F(\tilde{f}) = f$. We leave it to the reader to check directly that, for any functor $G : C \to D$, the canonical functor

$$B \times_D^{\text{strict}} C \to B \times_D C$$

$$(b, c) \mapsto (b, c, \text{id})$$

from the strict pullback in **Groupoids** to the homotopy pullback in **Groupoids** (with the effect on morphisms defined in the obvious way) is an equivalence of groupoids. We define a morphism of stacks to be a fibration if it is a fibration pointwise, so that strict pullbacks along fibrations in **Stacks** also coincide with the corresponding homotopy pullbacks, up to equivalence.

Definition 2.1.30. A morphism of stacks $f : \mathcal{M} \to \mathcal{N}$ is said to be *affine* if for any map $\mathrm{Spec}(R) \to \mathcal{N}$, the pullback $\mathcal{P}$ in the following diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] & \mathcal{N} \end{tikzcd}$$

is equivalent to an affine scheme.

Definition 2.1.31. For an affine morphism of stacks $f : \mathcal{M} \to \mathcal{N}$, we say f has algebraic geometry property P (e.g. flat, etale, surjective, open substack, G-torsor, etc.) if, for every pullback diagram

$$\begin{tikzcd} \mathrm{Spec}(S) \arrow[r] \arrow[d, "f'"'] & \mathcal{M} \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r] & \mathcal{N} \end{tikzcd}$$

f' has property P as a map between affine schemes. When we refer to a morphism f having property P, we always mean that f is affine and has property P as above.

Example 2.1.32. Fix a finite group G, and consider the morphism $f : \mathrm{Spec}(k) \to BG$ classifying the trivial G-torsor over $\mathrm{Spec}(k)$, i.e. this classifies the unique morphism of k-schemes

$$\mathrm{Spec}(\prod_{g \in G} k) \to \mathrm{Spec}(k)$$

The morphism of stacks f is affine, and it is a G-torsor (the universal one). Given a pullback diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d, "f'"'] & \mathrm{Spec}(k) \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[r, "\phi"] & BG \end{tikzcd}$$

f' is the G-torsor that ϕ classifies. Notice $\mathcal{P}$ is a discrete stack in the sense of Definition 2.1.19 because $\mathrm{Spec}(R)$ and $\mathrm{Spec}(k)$ are, so we just need to show that the presheaf

$$\mathbf{Aff}^{op} \xrightarrow{\mathcal{P}} \mathbf{Groupoids} \xrightarrow{\text{objects}} \mathbf{Sets}$$

is represented by the G-torsor $\phi \in \mathrm{Hom}_{\mathbf{Stacks}}(\mathrm{Spec}(R), BG) = BG(\mathrm{Spec}(R))$. Tracing thru the definition of the pullback, a point in $\mathcal{P}(\mathrm{Spec}(S))$ consists of a map

$$g : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$$

along with a trivialization of the G-torsor classified by $\phi \circ g$. A trivialization of a G-torsor is the same thing as a section of a G-torsor, so this data is the same as a that of a map $\mathrm{Spec}(S) \to T$, where T is the total space of the G-torsor classified by ϕ. Therefore $\mathcal{P} \simeq \mathrm{Hom}_{\mathbf{Aff}}(-, T)$.

Example 2.1.33. Let $\mathcal{M}$ be any stack. Define $\mathcal{M}_{(p)} = \mathcal{M} \times \mathrm{Spec}(k_{(p)})$ - this is a product of functors, i.e. pointwise product. There is a map $\iota : \mathcal{M}_{(p)} \to \mathcal{M}$ given by

$$\mathcal{M} \times \mathrm{Spec}(k_{(p)}) \to \mathcal{M} \times \mathrm{Spec}(k) \cong \mathcal{M}$$

This morphism is affine and exhibits $\mathcal{M}_{(p)}$ as an open substack of $\mathcal{M}$. To see ι is affine suppose we have a pullback diagram of stacks

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{M}_{(p)} \arrow[d, "\iota"] \\ \mathrm{Spec}(R) \arrow[r, "f"] & \mathcal{M} \end{tikzcd}$$

We want to know that $\mathcal{P}$ is equivalent to an affine scheme, and it's straightforward to show that it's equivalent to $\mathrm{Spec}(R) \times \mathrm{Spec}(k_{(p)}) \cong \mathrm{Spec}(R_{(p)})$. $\mathrm{Spec}(R_{(p)})$ is an open subscheme of $\mathrm{Spec}(R)$, so ι is an open immersion.

We move now to locally presentable stacks, which will comprise most of our examples.

Definition 2.1.34. A stack $\mathcal{M}$ is said to be *locally presentable* if the diagonal morphism

$$\mathcal{M} \xrightarrow{\Delta} \mathcal{M} \times \mathcal{M}$$

is affine.

Lemma 2.1.35.

1. *If $\mathcal{M}$ is a locally presentable stack, any morphism $\mathrm{Spec}(R) \to \mathcal{M}$ is affine.*

2. *If $\mathcal{M}$ is locally presentable, and $\mathcal{N} \subset \mathcal{M}$ is any substack, then $\mathcal{N}$ is locally presentable.*

Proof. Given a pullback diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathrm{Spec}(R) \arrow[d, "f"] \\ \mathrm{Spec}(S) \arrow[r, "g"] & \mathcal{M} \end{tikzcd}$$

one has a pullback diagram

$$\begin{tikzcd}
\mathcal{P} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\Delta"] \\
\operatorname{Spec}(R) \times \operatorname{Spec}(S) \arrow[r, "f \times g"] & \mathcal{M} \times \mathcal{M}
\end{tikzcd}$$

$\mathcal{P}$ is therefore affine, and this proves (1).

For (2), one checks that the square

$$\begin{tikzcd}
\mathcal{N} \arrow[r] \arrow[d, "\Delta"] & \mathcal{M} \arrow[d, "\Delta"] \\
\mathcal{N} \times \mathcal{N} \arrow[r] & \mathcal{M} \times \mathcal{M}
\end{tikzcd}$$

is a pullback; now observe that the pullback of an affine morphism is an affine morphism. $\square$

A large class of examples of locally presentable stacks come from so-called *Hopf Algebroids*. This is, in the first place, how stacks enter into chromatic homotopy, hence we place additional emphasis here.

Definition 2.1.36. A *groupoid* object in a category $\mathcal{C}$ is a pair (X_0, X_1) of objects in $\mathcal{C}$ along with morphisms

$$\begin{tikzcd}
X_0 \arrow[r, "\mathrm{id}"] & X_1 \arrow[l, bend right, "\text{codomain}"'] \arrow[l, bend left, "\text{domain}"]
\end{tikzcd}$$

and

$$X_1 \times_{X_0} X_1 \xrightarrow{\text{composition}} X_1$$

So that the functor

$$(\operatorname{Hom}_{\mathcal{C}}(-, X_0), \operatorname{Hom}_{\mathcal{C}}(-, X_1)) : \mathcal{C} \to \mathbf{Sets} \times \mathbf{Sets}$$

lifts to **Groupoids** along the functor

$$\mathbf{Groupoids} \xrightarrow{\text{(objects,morphisms)}} \mathbf{Sets} \times \mathbf{Sets}$$

via the given structure maps.

Along the equivalence $\mathbf{CAlg}_k \simeq \mathbf{Aff}^{op}$, a groupoid object $(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma))$ in $\mathbf{Aff}$ corresponds to what one may call a co-groupoid object in $\mathbf{CAlg}_k$.

Definition 2.1.37. A *Hopf Algebroid* over k is a pair of commutative k-algebras (A, Γ) along with morphisms in $\mathbf{CAlg}_k$

$$\begin{tikzcd} A \arrow[r, bend left, "\eta_R"] \arrow[r, bend right, "\eta_L"'] & \Gamma \arrow[l, "\epsilon"'] \end{tikzcd}$$

and

$$\Gamma \xrightarrow{\Delta} \Gamma \otimes_A \Gamma$$

such that η_L is flat, and such that this data determines a groupoid object structure on $(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma))$ in $\mathbf{Aff}$.

Remark 2.1.38. Since the functor

$$(\mathrm{Hom}_{\mathcal{C}}(-, X_0), \mathrm{Hom}_{\mathcal{C}}(-, X_1)) : \mathcal{C} \to \mathbf{Sets} \times \mathbf{Sets}$$

lifts not just to **Categories** but **Groupoids**, by the Yoneda lemma, one has the additional structure map $c : X_1 \xrightarrow{\text{inverse}} X_1$. The structure maps satisfy various coherence conditions corresponding to the defining properties of a groupoid, and these, in turn, determine coherences for the maps of k-algebras appearing in a Hopf algebroid (A, Γ). For instance, one has $c^2 = \mathrm{id}$ and $\eta_R = c \circ \eta_L$, so that flatness of η_L is equivalent to flatness of η_R. This flatness condition is not necessary for $(\mathrm{Spec}(A), \mathrm{Spec}(\Gamma))$ to have the structure of a groupoid object; it is included so that the category of comodules over (A, Γ) has the structure of an Abelian category, in particular so that it has kernels. For the reader interested in learning more about Hopf algebroids, the definitive source is Appendix A1 of [77].

Definition 2.1.39. Let (A, Γ) be a Hopf algebroid, then one has a prestack

$$\begin{tikzcd} & \mathbf{Groupoids} \arrow[d, "\text{(objects,morphisms)}"] \\ \mathbf{Aff}^{op} \arrow[ur, dashed, "\mathcal{M}^{pre}_{(A,\Gamma)}"] \arrow[r, "(\mathrm{Hom}_{\mathbf{Aff}}(-,\mathrm{Spec}(A))\,,\,\mathrm{Hom}_{\mathbf{Aff}}(-,\mathrm{Spec}(\Gamma)))"'] & \mathbf{Sets} \times \mathbf{Sets} \end{tikzcd}$$

We let $\mathcal{M}_{(A,\Gamma)}$ denote the stackification of $\mathcal{M}^{pre}_{(A,\Gamma)}$.

Example 2.1.40. Let (A, A) be the Hopf algebroid in which all structure maps are given by the identity map of A. The associated stack $\mathcal{M}_{(A,A)}$ is equivalent to the affine scheme $\mathrm{Spec}(A)$, where the latter is regarded as a stack taking values in discrete groupoids.

Example 2.1.41. *Quotient Stacks, revisited*: Let G be a finite group and R be a k-algebra with G-action. There is an equivalence of stacks

$$\mathrm{Spec}(R)/G \simeq \mathcal{M}_{(R, \prod\limits_{g \in G} R)}$$

where the former is as in Example 2.1.24, and the latter is given by the following Hopf algebroid structure:

- $\eta_L : R \to \prod\limits_{g \in G} R$ is given by the diagonal, and the g-th component of

  $$\eta_R : R \to \prod_{g \in G} R$$

  is given by the automorphism $g : R \to R$.

- The inversion map c is given by

  $$\prod_{g \in G} R \to \prod_{g \in G} R$$

  $$(r_g)_{g \in G} \mapsto (g \cdot r_{g^{-1}})_{g \in G}$$

- $\epsilon : \prod\limits_{g \in G} R \to R$ is projection onto the factor indexed by the identity element of G.

- The coproduct map

  $$\prod_{g \in G} R \to (\prod_{g \in G} R) \otimes_R (\prod_{g \in G} R)$$

  sends the basis element

  $$e_g = (0, \ldots, 1, \ldots, 0) \in \prod_{g \in G} R$$

  with its nonzero entry in the g-th component to the sum

  $$\Delta(e_g) = \sum_{\substack{g_1, g_2 \in G \\ g_1 \cdot g_2 = g}} e_{g_1} \otimes e_{g_2}$$

Proof. Referring to Example 2.1.24, the sheafification of the morphisms presheaf of the action groupoid

$$B_G(\mathrm{Hom}_{\mathbf{Aff}}(\mathrm{Spec}(R), -))$$

is $\mathrm{Spec}(\prod_{g \in G} R)$. One then uses the isomorphism of R-algebras

$$\prod_{g \in G} R \cong \mathrm{Map}(G, R)$$

to produce the above formulae. □

Remark 2.1.42. We pause to clarify how a Hopf algebroid (A, Γ) determines a functor valued in **Groupoids**. Let R be a commutative k-algebra, then $\mathrm{Hom}_{\mathbf{CAlg}_k}(A, R)$ forms the set of objects of a groupoid and $\mathrm{Hom}_{\mathbf{CAlg}_k}(\Gamma, R)$ forms the set of *all* morphisms (in particular, not just the set of morphisms between a particular pair of objects). For $f \in \mathrm{Hom}_{\mathbf{CAlg}_k}(\Gamma, R)$, one has a commutative diagram

$$\begin{tikzcd}
A \arrow[dr, "\eta_L"'] \arrow[drr, bend left, "c"] & & \\
& \Gamma \arrow[r, "f"] & R \\
A \arrow[ur, "\eta_R"] \arrow[urr, bend right, "d"'] & &
\end{tikzcd}$$

which represents the fact that f is a morphism from c to d. For composition, one has

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{CAlg}_k}(\Gamma, R) \times_{\mathrm{Hom}_{\mathbf{CAlg}_k}(A,R)} \mathrm{Hom}_{\mathbf{CAlg}_k}(\Gamma, R) \arrow[r, "\cong"] \arrow[dr] & \mathrm{Hom}_{\mathbf{CAlg}_k}(\Gamma \otimes_A \Gamma, R) \arrow[d, "\Delta^*"] \\
& \mathrm{Hom}_{\mathbf{CAlg}_k}(\Gamma, R)
\end{tikzcd}$$

where, for the first isomorphism, one uses the fact that the pushout in the category $\mathbf{CAlg}_k$ is the tensor product of commutative k-algebras.

For the rest of the chapter, we fix our site to be $(\mathbf{Aff}, \mathbf{Flat})$. We show now that, in the flat topology, a stack of the form $\mathcal{M}_{(A,\Gamma)}$ is the same thing as a locally presentable stack admitting a faithfully flat cover by an affine scheme. We begin with the following definition.

Definition 2.1.43. For a Hopf algebroid (A,Γ), let

$$p_A : \mathrm{Spec}(A) \to \mathcal{M}_{(A,\Gamma)}$$

denote the canonical map induced by the map of Hopf algebroids

$$(A,\Gamma) \to (A,A)$$

given by $\mathrm{id} : A \to A$ and $\epsilon : \Gamma \to A$, where the Hopf algebroid (A,A) is as in Example 2.1.40.

Theorem 2.1.44. *In the flat topology - i.e. in the category of stacks on* $(\mathbf{Aff},\mathbf{Flat})$ *- one has the following*

1. *For a Hopf algebroid* (A,Γ)*, the stack* $\mathcal{M}_{(A,\Gamma)}$ *is locally presentable, and the canonical map* $p_A : \mathrm{Spec}(A) \to \mathcal{M}_{(A,\Gamma)}$ *is a faithfully flat cover.*

2. *If* $\mathcal{M}$ *is a locally presentable stack, for any faithfully flat map* $p : \mathrm{Spec}(A) \to \mathcal{M}$*, a pullback diagram*

$$\begin{tikzcd} \mathrm{Spec}(\Gamma) \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathcal{M} \end{tikzcd}$$

determines the structure of a Hopf algebroid on the pair (A,Γ) *and induces an equivalence of stacks*

$$\mathcal{M}_{(A,\Gamma)} \xrightarrow{\simeq} \mathcal{M}$$

Proof. (cf. [44, Proposition 10.1 and Claim 10.4]) For (1), fix morphisms

$$f,g : \mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$$

and note that the following is a pullback square

$$\begin{tikzcd} \mathrm{Spec}(R) \times_{\mathcal{M}_{(A,\Gamma)}} \mathrm{Spec}(R) \arrow[r] \arrow[d] & \mathcal{M}_{(A,\Gamma)} \arrow[d, "\Delta"] \\ \mathrm{Spec}(R) \arrow[r, "(f,g)"] & \mathcal{M}_{(A,\Gamma)} \times \mathcal{M}_{(A,\Gamma)} \end{tikzcd}$$

It therefore suffices to show the pullback $\mathcal{P}$ in

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathrm{Spec}(R) \arrow[d, "g"] \\ \mathrm{Spec}(R) \arrow[r, "f"] & \mathcal{M}_{(A,\Gamma)} \end{tikzcd}$$

is equivalent to an affine scheme. We break this argument into the following smaller pieces:

- *Case I*: Let $f = g = p_A$ be the canonical map $\mathrm{Spec}(A) \to \mathcal{M}_{(A,\Gamma)}$. We claim the following is a pullback square

$$\begin{tikzcd} \mathrm{Spec}(\Gamma) \arrow[r, "\eta_R"] \arrow[d, "\eta_L"'] & \mathrm{Spec}(A) \arrow[d, "p_A"] \\ \mathrm{Spec}(A) \arrow[r, "p_A"] & \mathcal{M}_{(A,\Gamma)} \end{tikzcd}$$

  It is straightforward to show - using Remark 2.1.42 - that

$$\begin{tikzcd} \mathrm{Spec}(\Gamma) \arrow[r, "\eta_R"] \arrow[d, "\eta_L"'] & \mathrm{Spec}(A) \arrow[d, "p_A"] \\ \mathrm{Spec}(A) \arrow[r, "p_A"] & \mathcal{M}^{pre}_{(A,\Gamma)} \end{tikzcd}$$

  The result now follows from the fact that pullback commutes with stackification (see Remark 2.1.28), and that affine schemes are stacks.

- *Case II*: Let $f : \mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$ be a map factoring through the canonical map p_A and $g = p_A$. One has pullback squares

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathrm{Spec}(\Gamma) \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d, "g"] \\ \mathrm{Spec}(R) \arrow[r] \arrow[rr, bend right, "f"'] & \mathrm{Spec}(A) \arrow[r, "g"] & \mathcal{M}_{(A,\Gamma)} \end{tikzcd}$$

  Therefore, $\mathcal{P} \simeq \mathrm{Spec}(R \otimes_A \Gamma)$ and is, in particular, affine. We may argue similarly for the case in which $f, g : \mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$ both factor through the canonical map p_A.

- *Case III*: Let $f : \mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$ be any map and $g : \mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$ a map that factors thru p_A. Since $\mathcal{M}_{(A,\Gamma)}$ is a stack, there is a faithfully flat map

$$p : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$$

such that $f \circ p$ factors through p_A. By Case II, therefore, one has pullback squares

$$\begin{tikzcd}
\tilde{\mathcal{P}} \arrow[r] \arrow[d] & \mathcal{P} \arrow[r] \arrow[d] & \mathrm{Spec}(R) \arrow[d, "g"] \\
\mathrm{Spec}(S) \arrow[r, "p"] & \mathrm{Spec}(R) \arrow[r, "f"] & \mathcal{M}_{(A,\Gamma)}
\end{tikzcd}$$

where $\tilde{\mathcal{P}}$ is affine. This case then follows from Lemma 2.1.45 below.

- *Case IV*: Let $f, g : \mathrm{Spec}(R) \to \mathcal{M}_{(A,G)}$ be any maps, and choose a cover $p : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ such that $f \circ p$ factors through p_A. We have pullback squares

$$\begin{tikzcd}
\tilde{\mathcal{P}} \arrow[r] \arrow[d] & \mathcal{P} \arrow[r] \arrow[d] & \mathrm{Spec}(R) \arrow[d, "g"] \\
\mathrm{Spec}(S) \arrow[r, "p"] & \mathrm{Spec}(R) \arrow[r, "f"] & \mathcal{M}_{(A,\Gamma)}
\end{tikzcd}$$

where $\tilde{\mathcal{P}}$ is affine by Case III. Again $\mathcal{P}$ is affine by Lemma 2.1.45.

For (2), we must first show that the pullback diagram

$$\begin{tikzcd}
\mathrm{Spec}(\Gamma) \arrow[r, "\eta_R"] \arrow[d, "\eta_L"'] & \mathrm{Spec}(A) \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & \mathcal{M}
\end{tikzcd}$$

determines a Hopf algebroid structure on the pair (A, Γ). But by definition of pullback, the pair

$$(\mathrm{Hom}_{\mathbf{Aff}}(\mathrm{Spec}(R), \mathrm{Spec}(A)), \mathrm{Hom}_{\mathbf{Aff}}(\mathrm{Spec}(R), \mathrm{Spec}(\Gamma)))$$

is naturally identified with the groupoid $\mathcal{P}(\mathrm{Spec}(R))$ whose objects are maps $f : A \to R$ and whose morphisms $f_0 \to f_1$ are isomorphisms in $\mathcal{M}(\mathrm{Spec}(R))$

$$f_0^*(p_A) \to f_1^*(p_A)$$

where $p : \mathrm{Spec}(A) \to \mathcal{M}$ is the given cover. The existence and necessary properties of the Hopf algebroid structure maps then follow from the Yoneda lemma.

We have, moreover, a canonical morphism of stacks $\mathcal{M}^{pre}_{(A,\Gamma)} \to \mathcal{M}$ given pointwise by the obvious inclusion $\mathcal{P}(\mathrm{Spec}(R)) \to \mathcal{M}(\mathrm{Spec}(R))$. This is fully faithful and hence it remains so after applying stackification. It becomes essentially surjective after applying stackification: by passing to a faithfully flat cover, we can assume every object in $\mathcal{M}(\mathrm{Spec}(R))$ is of the form $f^*(p)$. $\square$

Lemma 2.1.45. *Suppose one has a pullback square of stacks*

$$\begin{tikzcd} \tilde{\mathcal{P}} \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\ \mathrm{Spec}(S) \arrow[r, "p"] & \mathrm{Spec}(R) \end{tikzcd}$$

such that $\tilde{\mathcal{P}}$ is affine and p is faithfully flat. Then $\mathcal{P}$ is affine.

Proof. Fix an equivalence $\mathrm{Spec}(A) \simeq \tilde{\mathcal{P}}$, then A is an S-module equipped with a descent datum with respect to the cover p. That is, there is an isomorphism of $S \otimes_R S$-modules $A \otimes_R S \to S \otimes_R A$ that satisfies the cocycle condition. This comes via universal property: let f and g be the maps in the following pullback square

$$\begin{tikzcd} \mathrm{Spec}(A \otimes_R S) \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{P} \arrow[d] \\ \mathrm{Spec}(S \otimes_R S) \arrow[r, "p \circ \pi_2"] & \mathrm{Spec}(R) \end{tikzcd}$$

Consider then the diagram

$$\begin{tikzcd} \mathrm{Spec}(A \otimes_R S) \arrow[dr, dashed] \arrow[drrr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & & \\ & \mathrm{Spec}(S \otimes_R A) \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\ & \mathrm{Spec}(S \otimes_R S) \arrow[r, "\pi_1"] & \mathrm{Spec}(S) \arrow[r, "p"] & \mathrm{Spec}(R) \end{tikzcd}$$

The larger rectangle is a pullback, and the dashed map is the desired isomorphism.

The dashed map is, in fact, an isomorphism of $S \otimes_R S$-algebras, hence we have a quasicoherent sheaf of algebras over $\mathrm{Spec}(S)$ equipped with a descent datum for the cover $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$. By faithfully flat descent for algebras, this sheaf must be pulled back from a quasicoherent sheaf of algebras on $\mathrm{Spec}(R)$, and the result follows (see [70, Theorem 2.23]). $\square$

2.1.3 Quasicoherent sheaves on stacks

In this subsection, we define (quasicoherent) sheaves on stacks and comodules over a Hopf algebroid. We show that when $\mathcal{M}$ is locally presentable and admits an affine cover that the category of quasicoherent sheaves over $\mathcal{M}$ is equivalent to the category of comodules over the corresponding Hopf algebroid. We continue to work in the flat site on affine schemes over k.

Definition 2.1.46. Suppose $\mathcal{M}$ is a stack on $(\mathbf{Aff}, \mathbf{Flat})$, we define the category $\mathbf{Aff}/\mathcal{M}$ as follows:

- An object of $\mathbf{Aff}/\mathcal{M}$ is a map $\mathrm{Spec}(A) \to \mathcal{M}$, i.e. a k-algebra A and a choice of $x_A \in \mathcal{M}(\mathrm{Spec}(A))$
- A morphism $(f, \phi) : x_A \to x_B$ is a morphism $f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ of k-algebras and a choice of isomorphism $\phi : x_A \to f^* x_B$ in $\mathcal{M}(\mathrm{Spec}(A))$, with composition defined in the obvious way.

Given a topology J on $\mathbf{Aff}$, we define the (big) J-site on $\mathcal{M}$ to be the category $\mathbf{Aff}/\mathcal{M}$, where a morphism (f, ϕ) is a covering if $f \in J$. We denote this Grothendieck site by $\mathbf{Aff}_J/\mathcal{M}$.

Remark 2.1.47. We have given the above definition for $\mathcal{M}$ a stack on $(\mathbf{Aff}, \mathbf{Flat})$ since that is our chosen setting, but there is no need for this. Note, in particular, that the stack $\mathcal{M}$ may be defined with respect to a different topology (that is, it satisfies descent with respect to a different topology) than the topology J used to define the site $\mathbf{Aff}_J/\mathcal{M}$.

Using the general formalism of Grothendieck sites, we can easily make sense of sheaf cohomology in this setting.

Definition 2.1.48. A *presheaf* $\mathcal{F}$ on a stack $\mathcal{M}$ is a presheaf of sets on the category $\mathbf{Aff}/\mathcal{M}$. A J*-sheaf* on $\mathcal{M}$ is a sheaf on the site $\mathbf{Aff}_J/\mathcal{M}$. An *abelian (pre)-sheaf* on $\mathcal{M}$ is a (pre)-sheaf on $\mathcal{M}$ valued in $\mathbf{Ab}$. A *(pre)-sheaf of rings* on $\mathcal{M}$ is a (pre)-sheaf on $\mathcal{M}$ valued in $\mathbf{CAlg}$.

Example 2.1.49. For a stack $\mathcal{M}$, we let the structure sheaf $\mathcal{O}_{\mathcal{M}}$ be defined by

$$\mathcal{O}_{\mathcal{M}}(\mathrm{Spec}(A) \to \mathcal{M}) = A$$

with the obvious functoriality. $\mathcal{O}_{\mathcal{M}}$ is an abelian sheaf on $\mathcal{M}$.

Definition 2.1.50. Let $\mathcal{F}$ be a presheaf on $\mathcal{M}$, we define the *global sections* of $\mathcal{F}$ to be the set

$$\Gamma(\mathcal{M};\mathcal{F}) \coloneqq \mathrm{Hom}_{\mathbf{Fun}((\mathbf{Aff}/\mathcal{M})^{op},\mathbf{Sets})}(*,\mathcal{F})$$

where here $*$ is the terminal object in $\mathbf{Fun}((\mathbf{Aff}/\mathcal{M})^{op},\mathbf{Sets})$. If $\mathcal{F}$ is an abelian presheaf, then $\Gamma(\mathcal{M};\mathcal{F})$ is naturally an abelian group, and we define the cohomology groups $H^i(\mathcal{M};\mathcal{F})$ to be the right derived functors of $\Gamma(\mathcal{M};-)$. See [85, tag 01FT] for more details.

We will be especially interested in the cohomology of quasicoherent sheaves for their relation to comodules in the locally presentable case. We define these now and remark on equivalent definitions one may find in the literature.

Definition 2.1.51. An $\mathcal{O}_{\mathcal{M}}$*-module* is an abelian sheaf $\mathcal{F}$ on $\mathcal{M}$ such that for each $x_A : \mathrm{Spec}(A) \to \mathcal{M}$, $\mathcal{F}(x_A)$ has the structure of an A-module, and for a morphism $(f,\phi) : x_A \to x_B$ in $\mathbf{Aff}/\mathcal{M}$, the map

$$\mathcal{F}(x_B) \to f^*\mathcal{F}(x_A)$$

is a map of B-modules, where $f^*\mathcal{F}(x_A)$ is the B-module given by restriction of scalars along f. An $\mathcal{O}_{\mathcal{M}}$*-algebra* is then defined in the obvious way.

We say an $\mathcal{O}_{\mathcal{M}}$-module $\mathcal{F}$ is *quasicoherent* if, for all $x_A \in \mathbf{Aff}_{/\mathcal{M}}$, the restricted sheaf $(x_A)^*\mathcal{F}$ is a quasicoherent $\mathcal{O}_{\mathrm{Spec}(A)}$-module.

Remark 2.1.52. Since quasicoherent sheaves on affine schemes $\mathrm{Spec}(R)$ are the same as modules over the ring of functions R, each restricted sheaf $(x_A)^*\mathcal{F}$ is simply a choice of A-module M_{x_A}. The functoriality with respect to $\mathbf{Aff}/\mathcal{M}$ guarantees that for every morphism $(f,\phi): x_A \to x_B$ in $\mathbf{Aff}/\mathcal{M}$, the corresponding map

$$\mathcal{F}(x_B) \to f^*\mathcal{F}(x_A)$$

of B-modules induces to an isomorphism of A-modules.

$$\mathcal{F}(x_A) \otimes_A B \to \mathcal{F}(x_A)$$

One can therefore use these conditions to define a quasicoherent sheaf on $\mathcal{M}$ in this way from a functor-of-points perspective, and this is the approach taken by Lurie in [58] and Goerss in [27].

Alternatively, one may define quasicoherent sheaves on $\mathcal{M}$ in the usual geometric way, by asking that there be some cover

$$p: \mathcal{U} \to \mathcal{M}$$

with the property that, for some sets I and J, there is an exact sequence

$$p^*\mathcal{O}_{\mathcal{M}}^{\oplus I}|_{\mathcal{U}} \to p^*\mathcal{O}_{\mathcal{M}}^{\oplus J}|_{\mathcal{U}} \to p^*\mathcal{F}|_{\mathcal{U}} \to 0$$

At least in the case when $\mathcal{U} \simeq \mathrm{Spec}(A)$ is affine and $\mathcal{M}$ is locally presentable (so that p is automatically an affine morphism), we have for any map $f: \mathrm{Spec}(R) \to \mathcal{M}$ a pullback square

$$\begin{tikzcd}
\mathrm{Spec}(\tilde{R}) \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d, "p"] \\
\mathrm{Spec}(R) \arrow[r, "f"] & \mathcal{M}
\end{tikzcd}$$

Since taking pullbacks of quasicoherent sheaves is right exact, the sequence remains exact when pulled back to $\mathrm{Spec}(\tilde{R})$, and the morphism $\mathrm{Spec}(\tilde{R}) \to \mathrm{Spec}(R)$ is a faithfully flat cover, so $f^*\mathcal{F}$ is a quasicoherent sheaf on $\mathrm{Spec}(R)$, and we recover Definition 2.1.51.

We finish this subsection by relating quasicoherent $\mathcal{O}_{\mathcal{M}}$-modules and algebras for $\mathcal{M}$ equivalent to a Hopf algebroid stack to comodules and comodule algebras over the Hopf algebroid.

Definition 2.1.53. Let (A,Γ) be a Hopf algebroid. A (left) *comodule* over (A,Γ) is an A-module M equipped with an A-linear coaction map

$$\psi : M \to \Gamma \otimes_A M$$

that is counitary and coassociative. A *comodule algebra* R over (A,Γ) is a commutative A-algebra that is also an (A,Γ)-comodule such that the coaction map is a map of A-algebras (see [77, A1.1.2] for more details).

Theorem 2.1.54. *There is an equivalence of abelian categories*

$$\mathbf{QCoh}(\mathcal{M}_{(A,\Gamma)}) \simeq \mathbf{Comod}_{(A,\Gamma)}$$

between the category of quasicoherent $\mathcal{O}_{\mathcal{M}_{(A,\Gamma)}}$-modules and the category of comodules over (A,Γ).

Proof. We begin by constructing a functor

$$\mathbf{QCoh}(\mathcal{M}_{(A,\Gamma)}) \to \mathbf{Comod}_{(A,\Gamma)}$$

Letting $p_A : \mathrm{Spec}(A) \to \mathcal{M}_{(A,\Gamma)}$ be the canonical cover, $\mathcal{F} \in \mathbf{QCoh}(\mathcal{M}_{(A,\Gamma)})$ determines an A-module

$$M := (p_A)^*\mathcal{F}$$

By functoriality with respect to the pullback diagram

$$\begin{tikzcd} \mathrm{Spec}(\Gamma) \arrow[r, "\eta_R"] \arrow[d, "\eta_L"] & \mathrm{Spec}(A) \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathcal{M}_{(A,\Gamma)} \end{tikzcd}$$

we have an isomorphism of Γ-modules

$$\phi : M \otimes_A \Gamma \to \Gamma \otimes_A M$$

M then becomes a left (A, Γ)-comodule via

$$M \to M \otimes_A \Gamma \xrightarrow{\phi} \Gamma \otimes_A M$$

$$m \mapsto m \otimes 1 \mapsto \phi(m \otimes 1)$$

Going the other way, if $(M, \psi) \in \mathbf{Comod}_{(A,\Gamma)}$, we define $(\mathcal{F}_M)(p_A) = M$ and for any map $f : \mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$ that factors through p_A, we define $(\mathcal{F}_M)(f)$ by pulling back M along the map $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$. Suppose now that $f : \mathrm{Spec}(R) \to \mathcal{M}_{(A,\Gamma)}$ is any map, and consider the following diagram:

$$\begin{tikzcd}
 & \mathrm{Spec}(\Gamma) \arrow[rr] \arrow[dd] & & \mathrm{Spec}(A) \arrow[dd, "p_A"] \\
\mathrm{Spec}(B \otimes_R B) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Spec}(B) \arrow[ur] \arrow[dd] & \\
 & \mathrm{Spec}(A) \arrow[rr] & & \mathcal{M}_{(A,\Gamma)} \\
\mathrm{Spec}(B) \arrow[ur] \arrow[rr] & & \mathrm{Spec}(R) \arrow[ur, "f"] &
\end{tikzcd}$$

where the front and back squares are pullbacks. To define an R-module $\mathcal{F}(f)$, note that the map $\mathrm{Spec}(B) \to \mathcal{M}_{(A,\Gamma)}$ factors through p_A, so we may define a B-module $M \otimes_A B$ associated to this composite as before. This B-module comes equipped with canonical descent datum for the cover $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$, by pulling back the descent datum for the cover $\mathrm{Spec}(A) \to \mathcal{M}_{(A,\Gamma)}$ given by the A module M and the isomorphism of Γ-modules

$$\Gamma \otimes_A M \xrightarrow{1 \otimes \psi} \Gamma \otimes_A \Gamma \otimes_A M \xrightarrow{\tau} M \otimes_A \Gamma \otimes_A \Gamma \xrightarrow{1 \otimes \mu} M \otimes_A \Gamma$$

where

$$\tau(g_1 \otimes g_2 \otimes m) = m \otimes g_1 \otimes g_2$$

and μ is the multiplication map on Γ. By faithfully flat descent this determines an R-module which we define to be $\mathcal{F}(f)$. We refer the reader to the proof of Theorem A in [46] for more details. $\square$

Remark 2.1.55. One has, as usual, a tensor product of $\mathcal{O}_{\mathcal{M}}$ modules given by sheafifying the presheaf

$$(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{G})^{pre}(\mathrm{Spec}(A) \xrightarrow{x_A} \mathcal{M}) = \mathcal{F}(x_A) \otimes_A \mathcal{G}(x_A)$$

Under the equivalence of the preceding theorem, one can show this corresponds to the comodule tensor product, where $(M, \psi_M) \otimes (N, \psi_N)$ is the A-module $M \otimes_A N$ with coaction

$$M \otimes_A N \xrightarrow{\psi_M \otimes \psi_N} \Gamma \otimes_A M \otimes_A \Gamma \otimes_A N \xrightarrow{1 \otimes \tau \otimes 1} \Gamma \otimes_A \Gamma \otimes_A M \otimes_A N \xrightarrow{\mu \otimes 1 \otimes 1} \Gamma \otimes_A M \otimes_A N$$

2.1.4 $\mathcal{M}_{FG}$

Chromatic homotopy enters into the world of algebraic geometry and stacks via the moduli stack of formal groups. We explain in the next section how this comes from topology; in this section we stay grounded in the algebraic geometry and prove some of the main results concerning the geometry of $\mathcal{M}_{FG}$. This stack turns out to be surprisingly rigid: we show that - when localized at a prime - the classical invariant prime ideal theorem gives a classification of the open substacks of $\mathcal{M}_{FG}$. This, along with Lazard's theorem on height, gives a complete description of the *associated space* of the stack $\mathcal{M}_{FG}$. This is a significant step in understanding a moduli problem. Going further, one seeks to understand the automorphisms of the points in this space, and this leads one to studying Morava stabilizer groups and their action on Lubin-Tate spaces. We do not discuss this, as will not need this part of the theory for our applications. For the reader interested in reading about this part of the theory from the stacks point of view, we recommend Lurie's notes [58] and Goerss's notes [27]. We continue in this section to work in the flat topology on affine schemes, and we fix our ground ring k to be $\mathbb{Z}$, or $\mathbb{Z}_{(p)}$ when working locally at a prime p. We begin by defining our arithmetic object of interest:

Definition 2.1.56. A *formal group law* over a commutative ring R is a power series

$$F(x, y) = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j \in R[[x, y]]$$

satisfying the following properties:

- $F(x,0) = x$
- $F(x,y) = F(y,x)$
- $F(F(x,y),z) = F(x,F(y,z)) \in R[[x,y,z]]$

Remark 2.1.57. These properties of $F(x,y)$ read like the defining properties of an abelian group, and that is no accident; indeed, a choice of formal group law F over R is equivalent to a choice of co-abelian group object structure on $R[[x]]$ in the category of topological R-algebras, where $R[[x]]$ has the (x)-adic topology. Working dually, the (x)-adic topology on $R[[x]]$ is replaced by working in the category of *formal schemes*. In particular, for a category $\mathcal{C}$ we may define the category $\mathbf{Ind}(\mathcal{C})$ to have objects sequences

$$\{c_0 \to c_1 \to c_2 \to \cdots\}$$

for $c_i \in \mathcal{C}$, such that

$$\mathrm{Hom}_{\mathbf{Ind}(\mathcal{C})}(\{c_i\},\{d_i\}) = \lim_i \operatorname{colim}_j \mathrm{Hom}_{\mathcal{C}}(c_i,d_j)$$

We then simply define the category of formal schemes over R to be

$$\mathbf{FormalSch}_{/R} \coloneqq \mathbf{Ind}(\mathbf{Aff}_{/\mathrm{Spec}(R)})$$

and a formal group law over R is the data of an abelian group structure on the formal R-scheme

$$\mathrm{Spf}(R[[x]]) \coloneqq \{\mathrm{Spec}(R) \to \mathrm{Spec}(R[x]/x^2) \to \mathrm{Spec}(R[x]/x^3) \to \cdots\}$$

Example 2.1.58. We give two straightforward examples of formal group laws. Both of these come from the same construction: there is a functor

$$\mathbf{AbGrp}(\mathbf{Aff}_{/\mathrm{Spec}(R)}) \to \mathbf{AbGrp}(\mathbf{FormalSch}_{/R})$$

between categories of abelian group objects which is called "completion at the identity." If $\mathbb{G} \cong \mathrm{Spec}(S)$ is an affine abelian group scheme over $\mathrm{Spec}(R)$, one has an R-algebra map

$\epsilon : S \to R$ dual to the identity structure map. One checks that the group structure on $\mathbb{G}$ determines a group structure on the formal R-scheme

$$\widehat{\mathbb{G}} := \{\mathrm{Spec}(R) \to \mathrm{Spec}(S/\ker(\epsilon)^2) \to \mathrm{Spec}(S/\ker(\epsilon)^3) \to \cdots\}$$

Given an isomorphism of formal schemes $\widehat{\mathbb{G}} \cong \mathrm{Spf}(R[[x]])$, one therefore has a formal group law over R.

- $\mathbb{G}_a$: Consider the group scheme defined by $\mathbb{G}_a(S) = S$ for an R-algebra S, where S is regarded as an abelian group by forgetting the ring structure. One has an isomorphism $\mathbb{G}_a \cong \mathrm{Spec}(R[x])$, where the group structure is dual to the map

$$R[x] \to R[x] \otimes_R R[x] \cong R[x, y]$$
$$x \mapsto x \otimes 1 + 1 \otimes x \mapsto x + y$$

  and the identity map $\epsilon : R[x] \to R$ is the one sending x to zero, so that $\ker(\epsilon) = (x)$. It follows that $\widehat{\mathbb{G}_a}$ is determined by the formal group law $F(x, y) = x + y$.

- $\mathbb{G}_m$: Consider the group scheme defined by $\mathbb{G}_a(S) = S^\times$ for an R-algebra S. One has an isomorphism $\mathbb{G}_m \cong \mathrm{Spec}(R[x^\pm])$, where the group structure is dual to the map

$$R[x^\pm] \to R[x^\pm] \otimes_R R[x^\pm] \cong R[x^\pm, y^\pm]$$
$$x \mapsto x \otimes x \mapsto xy$$

  and the identity map $\epsilon : R[x^\pm] \to R$ is the one sending x to 1, so that $\ker(\epsilon) = (x - 1)$. One has isomorphisms

$$R[x^\pm]/\ker(\epsilon)^n \cong R[x^\pm]/(x-1)^n \cong R[t]/t^n$$

  where the latter isomorphism sends $x \mapsto t + 1$. It follows that $\widehat{\mathbb{G}_m}$ is determined by the formal group law $F(t_1, t_2) = t_1 + t_2 + t_1 t_2$.

We are of course interested in the moduli problem of such group laws, so we make the following definition:

Definition 2.1.59. An *isomorphism of formal group laws* $f : F_1 \to F_2$ over R is a power series

$$f(x) = b_0 x + b_1 x^2 + b_2 x^3 + \cdots \in R[[x]]$$

such that $b_0 \in R^\times$ and $f(F_1(x,y)) = F_2(f(x), f(y)) \in R[[x,y]]$. We say the isomorphism f is *strict* if $b_0 = 1$.

The condition that b_0 be a unit guarantees that the power series f has a compositional inverse, as may be checked explicitly. This defines a groupoid of formal group laws over a ring R, and this groupoid is functorial in R in a natural way. Suppose $f : R \to S$ is a ring map, and

$$F(x,y) = x + y + \sum_{i,j} a_{ij} x^i y^j$$

is a formal group law over R. We may push forward F along f, setting

$$f^* F(x,y) = x + y + \sum_{i,j \geq 1} f(a_{ij}) x^i y^j$$

and it follows that $f^* F$ is a formal group law over S. We use the pullback notation f^* because this construction is the natural pullback construction on the corresponding formal schemes. The following theorem gives us substantial control over this moduli problem (see Lazard [55] and [77, Appendix A2]).

Theorem 2.1.60. *The functor* **CAlg** $\to$ **Sets** *sending a commutative ring R to the set of formal group laws over R is corepresentable. The co-representing ring L is called the Lazard ring and there is an isomorphism*

$$L \cong \mathbb{Z}[x_1, x_2, \ldots]$$

The groupoid of formal group laws and isomorphisms as in Definition 2.1.59 determines a Hopf algebroid structure on the pair $(L, L[b_0^{\pm}, b_1, b_2, \ldots])$, and the groupoid of formal group laws and strict isomorphisms determines a Hopf algebroid $(L, L[b_1, b_2, \ldots])$.

Definition 2.1.61. We define the stacks

$$\mathcal{M}_{FG} := \mathcal{M}_{(L, L[b_0^{\pm}, b_1, b_2, \ldots])}$$

$$\mathcal{M}_{FG}(1) := \mathcal{M}_{(L, L[b_1, b_2, \ldots])}$$

Remark 2.1.62. We pause to comment on gradings. We define a graded ring R_* to be an associative monoid in the category of graded abelian groups with the graded tensor product; a graded-commutative ring R_* is a commutative monoid, where the braiding is given by the usual Koszul sign rule. We define, on the other hand, a commutative graded ring to be a commutative ring R along with an isomorphism of abelian groups

$$R \cong \bigoplus_{i \in \mathbb{Z}} R_i$$

such that $1 \in R_0$ and the multiplication on R restricts to maps $R_i \otimes R_i \to R_{i+j}$. Note that when a graded-commutative ring R_* is concentrated in even degrees, setting $R = \bigoplus_{i \in \mathbb{Z}} R_i$ gives a commutative graded ring.

For a commutative ring R, the structure of a commutative graded ring on R is equivalent to an action of the affine group scheme $\mathbb{G}_m$ on the affine scheme $\mathrm{Spec}(R)$. In particular, such an action is equivalent to the structure of a $\mathbb{Z}[x^{\pm}]$-comodule algebra on R, where the Hopf algebra $\mathbb{Z}[x^{\pm}]$ is defined by setting $\Delta(x) = x \otimes x$, and the homogeneous elements $r \in R$ of degree n are those such that $\psi(r) = x^r \otimes r$, where ψ is the coaction map

$$R \xrightarrow{\psi} \mathbb{Z}[x^{\pm}] \otimes R$$

In the above theorem, setting $|x_i| = 2i$ and $|b_i| = 2i$ determines a $\mathbb{G}_m$ action on the stack $\mathcal{M}_{FG}(1)$. This action can also be described as the one sending a unit $r \in R^{\times}$ and a formal group law $F(x, y)$ over R to the formal group law

$$r^{-1} F(rx, ry)$$

Using this, one may check that there is an equivalence of stacks

$$\mathcal{M}_{FG}(1)/\mathbb{G}_m \simeq \mathcal{M}_{FG}$$

and the $\mathbb{G}_m$-torsor $\mathcal{M}_{FG}(1) \to \mathcal{M}_{FG}$ is classified by the line bundle $\omega : \mathcal{M}_{FG} \to B\mathbb{G}_m$ discussed below in Remark 2.1.67.

We have only defined quotient stacks $\mathcal{M}/\mathbb{G}_m$ when $\mathcal{M}$ is an affine scheme. However, there is a straightforward definition that resembles that of the homotopy orbit construction on G-spaces. For a group scheme G, we set

$$\mathcal{M}/G := {}^{a}(EG \times_G \mathcal{M})$$

where EG is the prestack sending $\mathrm{Spec}(R)$ to the action groupoid $B_{G(R)}(G(R))$ of the $G(R)$-set $G(R)$.

Definition 2.1.63. There is more generally a stack $\mathcal{M}_{FG}(m)$, the moduli stack of formal groups together with an m-jet. That is,

$$\mathcal{M}_{FG}(m)(R) = \begin{cases} \textit{Objects}\text{: formal group laws over } R \\ \textit{Morphisms: } f : F \to G \text{ such that } f(x) \equiv x \mod x^{m+1} \end{cases}$$

These stacks will be especially important to us in the final chapter.

We show now that the functor sending a commutative ring R to the groupoid of formal group laws over R and *strict* isomorphisms satisfies descent: i.e. one has an equivalence

$$\mathcal{M}^{pre}_{(L, L[b_1, b_2, \ldots])} \simeq \mathcal{M}_{FG}(1)$$

Theorem 2.1.64. *The prestack $\mathcal{M}^{pre}_{(L, L[b_1, b_2, \ldots])}$ is a stack.*

Proof. $\mathrm{Spec}(L)$ and $\mathrm{Spec}(L[b_1, b_2, \ldots])$ are representable presheaves and therefore sheaves, as the flat topology is subcanonical (see [85, tag 03NV]). We set

$$\mathcal{M} := \mathcal{M}^{pre}_{(L, L[b_1, b_2, \ldots])}$$

It remains to show, therefore, that for every faithfully flat cover

$$\coprod_i \mathrm{Spec}(R_i) \to \mathrm{Spec}(R)$$

whenever we have $F_i \in \mathcal{M}(\mathrm{Spec}(R_i))$ along with isomorphisms

$$f_{ij} : F_i|_{\mathrm{Spec}(R_i)\times_{\mathrm{Spec}(R)}\mathrm{Spec}(R_j)} \to F_j|_{\mathrm{Spec}(R_i)\times_{\mathrm{Spec}(R)}\mathrm{Spec}(R_j)}$$

satisfying the cocycle condition (i.e. $f_{ik} = f_{jk} \circ f_{ij}$), there exists a formal group law

$$G \in \mathcal{M}(\mathrm{Spec}(R))$$

so that $G|_{\mathrm{Spec}(R_i)} \cong F_i$ for all i. We set $X = \mathrm{Spec}(R)$ and $U_i = \mathrm{Spec}(R_i)$.

For this argument, we need to recall the basics of Cech cohomology. Suppose $\mathcal{F}$ is a sheaf, and $\{U_i \to X\}$ is a cover, then we may form the Cech nerve of this cover, a simplicial object:

$$\coprod_i U_i \leftleftarrows \coprod_{i,j} U_i \times_X U_j \Lleftarrow \coprod_{i,j,k} U_i \times_X U_j \times_X U_k \cdots$$

Applying $\mathcal{F}$ to this diagram, we have a cosimplicial object

$$\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_X U_j) \Rrightarrow \prod_{i,j,k} \mathcal{F}(U_i \times_X U_j \times_X U_j) \cdots$$

If $\mathcal{F}$ is an abelian sheaf, we can take the alternating sum of the maps appearing in this cosimplicial object, and we get a cochain complex. This cochain complex, by definition, computes the Cech cohomology of $\mathcal{F}$ with respect to the given cover $\{U_i \to U\}$. The "cocycle condition" that we see on a family α_{ij} then becomes exactly the condition that the tuple (α_{ij}) forms a cocycle in this chain complex, in particular an element of $\check{Z}^1$, and so it represents a class in $\check{H}^1(X; \mathcal{F})$. That condition simply reads

$$\alpha_{ik} = \alpha_{jk} + \alpha_{ij}$$

for all i, j, k. When $\mathcal{F} = \mathbb{G}_a$, this cohomology group *always* vanishes on affine schemes (in both the Zariski and flat topologies). $\mathbb{G}_a$ is quasicoherent, and X is affine, so this follows from Serre's vanishing theorem (see [85, tag 03OY]).

Since the functor $R \mapsto \{\text{Formal Group Laws over } R\}$ is a sheaf of sets, if instead of the (F_i, f_{ij}) we have above, we simply had G_i such that $G_i|_{U_i \times_X U_j}$ and $G_j|_{U_i \times_X U_j}$ were *equal*, then we could construct the desired G restricting to G_i, by the gluing condition of a sheaf. We are of course given less than that, but we're going to use the above observations about Cech cohomology to show that we can change our F_i's up to isomorphism to form a family G_i as above.

We proceed by induction; we construct a family of power series $g_i(x)$ inductively, and our G_i's will be conjugates of the F_i's by these g_i's. Setting

$$g_i^{(1)}(x) = x$$

for all i, we assume by induction there exists a family of power series

$$g_i^{(n-1)}(x) \in \mathcal{O}_X(U_i)[[x]] = R_i[[x]]$$

with the property that

$$(g_j^{(n-1)})((g_i^{(n-1)})^{-1}(x)) \equiv f_{ij}(x) \mod x^n$$

as power series in x over the ring $\mathcal{O}_X(U_i \times_X U_j) = R_i \otimes_R R_j$. Now we will define

$$g_i^{(n)}(x) = g_i^{(n-1)}(x) + b_n(i)x^n$$

and we will show that we can choose $b_n(i) \in R_i$ appropriately. We make use of the following fact: if $f(x)$ is some invertible power series, then if we set $g(x) = f(x) + ax^n$,

$$g^{-1}(x) \equiv f^{-1}(x) - ax^n \mod x^{n+1}$$

This gives

$$\begin{aligned}
&(g_j^{(n)})((g_i^{(n)})^{-1}(x)) \\
&\qquad \equiv g_j^{(n-1)}((g_i^{(n-1)})^{-1}(x) - b_n(i)x^n) + b_n(j)(g_i^{(n-1)}(x) - b_n(i)x^n)^n \mod x^{n+1} \\
&\qquad \equiv g_j^{(n-1)}((g_i^{(n-1)})^{-1}(x)) + (b_n(j) - b_n(i))x^n \mod x^{n+1}
\end{aligned}$$

but by our inductive hypothesis, we have

$$g_j^{(n-1)}((g_i^{(n-1)})^{-1}(x)) \equiv f_{ij}(x) + c_{ij}^{(n)}x^n \mod x^{n+1}$$

for some $c_{ij}^{(n)}$. The induction is complete if we can choose a family $b_n(i)$'s with the property that $b_n(i)-b_n(j) = c_{ij}^{(n)}$, i.e. if the family $c_{ij}^{(n)}$ is a coboundary in the Cech complex computing $\check{H}^1(X;\mathbb{G}_a) = 0$. It is therefore sufficient to know that the $c_{ij}^{(n)}$'s form a cocycle. This amounts to knowing that $c_{ik}^{(n)} = c_{ij}^{(n)} + c_{jk}^{(n)}$ for all i,j,k on the triple intersections $\mathcal{O}_X(U_i \times_X U_j \times_X U_k)$, but we know that $f_{ik} = f_{jk} \circ f_{ij}$ so we have

$$\begin{aligned}
&g_k^{(n-1)}((g_i^{(n-1)})^{-1}(x)) - c_{ik}^{(n)}x^n \\
&\qquad \equiv f_{ik}(x) \mod x^{n+1} \\
&\qquad \equiv f_{jk}(f_{ij}(x)) \mod x^{n+1} \\
&\qquad \equiv f_{jk}(g_j^{(n-1)}((g_i^{(n-1)})^{-1}(x)) - c_{ij}^{(n)}x^n) \mod x^{n+1} \\
&\qquad \equiv f_{jk}(g_j^{(n-1)}((g_i^{(n-1)})^{-1}(x))) - c_{ij}^{(n)}x^n \mod x^{n+1} \\
&\qquad \equiv g_k^{(n-1)}((g_j^{(n-1)})^{-1}(g_j^{(n-1)}((g_i^{(n-1)})^{-1}(x)))) - c_{jk}^{(n)}(g_j^{(n-1)}((g_i^{(n-1)})^{-1}(x)))^n \\
&\qquad\quad - c_{ij}^{(n)}x^n \mod x^{n+1} \\
&\qquad \equiv g_k^{(n-1)}((g_i^{(n-1)})^{-1}(x)) - c_{jk}^{(n)}x^n - c_{ij}^{(n)}x^n \mod x^{n+1}
\end{aligned}$$

from which the result follows by collecting the coefficients of x^n on each side.

We now define

$$G_i(x,y) = g_i^{-1}(F_i(g_i(x), g_i(y)))$$

All that's left to check is that

$$G_i|_{U_i \times_X U_j} = G_j|_{U_i \times_X U_j}$$

But we have

$$\begin{aligned}
g_j(G_i(x,y)) &= g_j(g_i^{-1}(F_i(g_i(x), g_i(y)))) \\
&= f_{ij}(F_i(g_i(x), g_i(y))) \\
&= F_j(f_{ij}(g_i(x), g_i(y))) \\
&= F_j(g_j(x), g_j(y))
\end{aligned}$$

where we used $f_{ij} = g_j g_i^{-1}$ and that f_{ij} is an isomorphism from F_i to F_j. Applying g_j^{-1} to both sides we have

$$G_i(x,y) = g_j^{-1}(F_j(g_j(x), g_j(y))) = G_j(x,y)$$

□

Remark 2.1.65. The above argument is elementary and explicit, but one can summarize the argument in a more conceptual way (see also [58, Lecture 11]). The difficulty in this approach is that one has to define $\check{H}^1$ for a sheaf of nonabelian groups; this is not difficult, but we omit it here. Suppose there is a reasonable notion of $\check{H}^1(X;\mathbb{G})$ for a sheaf of nonabelian groups $\mathbb{G}$ with the property that it recovers the usual notion when $\mathbb{G}$ is abelian, sends short exact sequences of group schemes to exact in the middle sequences, and cocycle data as in our f_{ij}'s determine classes in $\check{H}^1$, so that they determine a coboundary precisely when these g_i can be constructed. If $\mathbb{G}$ is the group scheme represented by $\operatorname{Spec}(\mathbb{Z}[b_1, b_2, \ldots])$ with group structure coming from composition of power series $x + \sum_{i\geq 1} b_i x^{i+1}$, then $\mathbb{G}$ has subgroup schemes $\mathbb{G}_n$, whose R-points consist of power series of the form

$$x + b_n x^{n+1} + \cdots$$

i.e. those that are congruent to x mod x^{n+1}. The R-points of the quotient group scheme $\mathbb{G}/\mathbb{G}_n$ can be identified with the subgroup of "strict" automorphisms of the affine formal scheme

$$\{\operatorname{Spec}(R) \to \operatorname{Spec}(R[x]/x^2) \to \operatorname{Spec}(R[x]/x^3) \to \cdots \to \operatorname{Spec}(R[x]/x^n) \xrightarrow{=} \cdots\}$$

More explicitly, $\mathbb{G}/\mathbb{G}_n$ is Spec of the ring that corepresents polynomials of the form

$$x + b_1 x^2 + \cdots + b_{n-2} x^{n-1}$$

with group structure given by composition according to the rule that $x^n = 0$. Since every power series is uniquely determined by its truncations mod x^n for every n, it follows that $\mathbb{G}$

is the inverse limit of $\mathbb{G}/\mathbb{G}_n$. We therefore need only show that $\check{H}^1(X;\mathbb{G}/\mathbb{G}_n)$ vanishes, for all n. Noting that $\mathbb{G}/\mathbb{G}_2 \cong \mathbb{G}_a$ and $\mathbb{G}_n/\mathbb{G}_{n+1} \cong \mathbb{G}_a$, the short exact sequences coming from $\mathbb{G}_{n+1} \subset \mathbb{G}_n \subset \mathbb{G}$

$$1 \to \mathbb{G}_n/\mathbb{G}_{n+1} \to \mathbb{G}/\mathbb{G}_{n+1} \to \mathbb{G}/\mathbb{G}_n \to 1$$

tell us by induction that since $\check{H}^1(X;\mathbb{G}_a) = 0$, $\check{H}^1(X;\mathbb{G}/\mathbb{G}_n) = 0$ for all n.

In fact, the conclusion of Theorem 2.1.64 is true for $\mathcal{M}_{FG}(m)$ for all $m \geq 1$. This is not the case for $\mathcal{M}_{FG}$ in the non-strict case. The above argument breaks down in the base case; namely, we can't simply define $g_i^{(1)}(x) = x$ because we aren't assuming that our isomorphisms f_{ij} are strict. If we wanted to run the same argument, since the b_0's have to be units, we would at the very first step encounter the group

$$\check{H}^1(X;\mathbb{G}_m)$$

In contrast to $\mathbb{G}_a$, these cohomology groups don't always vanish - in fact they are isomorphic to the group of $\mathbb{G}_m$-torsors on X, or equivalently line bundles on X. These cohomology classes are thus the obstructions to $(\mathcal{M}_{FG})^{pre}$ being a stack. The question becomes - how does one describe the objects in $\mathcal{M}_{FG}(\mathrm{Spec}(R))$? That is, what happens to formal group laws upon stackification? We make the following somewhat flippant definition and then use our understanding of stackification to elaborate.

Definition 2.1.66. A *formal group* over a commutative ring R is an object of the groupoid $\mathcal{M}_{FG}(\mathrm{Spec}(R))$.

Remark 2.1.67. In principle, a formal group over R is then simply a choice of faithfully flat cover $\{\mathrm{Spec}(R_i) \to \mathrm{Spec}(R)\}_{i\in I}$, a choice of formal group law F_i on R_i for each i, and a choice of family of isomorphisms

$$f_{ij} : F_i|_{\mathrm{Spec}(R_i)\times_{\mathrm{Spec}(R)}\mathrm{Spec}(R_j)} \to F_j|_{\mathrm{Spec}(R_i)\times_{\mathrm{Spec}(R)}\mathrm{Spec}(R_j)}$$

that satisfy the cocycle condition. The various b_0's that appear in the power series f_{ij}'s then determine a line bundle on $\mathrm{Spec}(R)$ as above, and this line bundle is what we call ω. Since

ω is natural under pullbacks, it determines a line bundle on $\mathcal{M}_{FG}$, i.e. a quasicoherent sheaf $\mathcal{F}$ on $\mathcal{M}_{FG}$ with the property that for all maps

$$x_A : \mathrm{Spec}(A) \to \mathcal{M}_{FG}$$

$(x_A)^*\mathcal{F}$ is a line bundle on $\mathrm{Spec}(A)$.

This definition, however, is somewhat clunky, so we make use of the following proposition:

Proposition 2.1.68. *(See [44, Definition 15.5]) A formal group over R is equivalent data to the following*

- *An augmented R-algebra $\epsilon : A \to R$ with augmentation ideal $\mathfrak{m}$ such that*

  1. *A is complete with respect to the $\mathfrak{m}$-adic topology.*
  2. *$\mathfrak{m}/\mathfrak{m}^2$ is locally free of rank one over R.*
  3. *The associated graded of the $\mathfrak{m}$-adic filtration of A is isomorphic to $\mathrm{Sym}_R(\mathfrak{m}/\mathfrak{m}^2)$.*

- *An abelian group object structure on the formal R-scheme $\{\mathrm{Spec}(A/\mathfrak{m}^n)\}$.*

Proof. Given a formal group $\mathbb{G}$ over R, the line bundle ω from the previous remark determines a locally free of rank one R-module P. Let A be the commutative ring

$$R[[P]] := \varprojlim_n R[P]/P^n$$

where $R[P] = \mathrm{Sym}_R(P)$ and $R[P]/P^n$ is the quotient of $R[P]$ by the n-th power of its augmentation ideal. The group structures on

$$\mathrm{Spf}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R_i) \cong \mathrm{Spf}(R_i[[x]])$$

glue to give $\mathrm{Spf}(A)$ an abelian group structure. Conversely, if we are given the structure in the proposition, pulling back to each $\mathrm{Spec}(R_i)$, one has a family formal group laws with descent data, which determines an object of $\mathcal{M}_{FG}(\mathrm{Spec}(R))$ since $\mathcal{M}_{FG}$ is a stack. $\square$

Remark 2.1.69. We have defined the stack $\mathcal{M}_{FG}$ in the flat topology. It turns out that for $\mathcal{M}_{FG}$, we could have used the Zariski topology and nothing would change. This is because the only obstruction to stackification is this line bundle ω above. In both the Zariski and flat topologies, a line bundle over an affine scheme is the same thing as a projective module of rank one (see [44, 15.7]). This is why, for instance, in the definition of formal group appearing in [58, Lecture 11], Lurie uses Zariski covers.

Definition 2.1.70. A *coordinate* on a formal group $\mathbb{G}$ over R is a choice of formal group law F over R and an isomorphism $F \to \mathbb{G}$ in $\mathcal{M}_{FG}(\mathrm{Spec}(R))$. Equivalently, a coordinate on $\mathbb{G}$ is a choice of isomorphism of formal R-schemes

$$\mathbb{G} \cong \mathrm{Spf}(R[[x]])$$

Remark 2.1.71. Note that a coordinate on $\mathbb{G}$ exists if and only if the line bundle ω associated to $\mathbb{G}$ is trivializable. We summarize this in the following tower of fibrations, which shows also how the stacks $\mathcal{M}_{FG}(m)$ fit into the cohomological argument used in Theorem 2.1.64

$$\begin{tikzcd}
 & \mathrm{Spec}(L) \arrow[r] \arrow[d] & B\mathbb{G}_a \\
 & \vdots \arrow[d] & \\
 & \mathcal{M}_{FG}^{(n)} \arrow[r] \arrow[d] & B\mathbb{G}_a \\
 & \vdots \arrow[d] & \\
 & \mathcal{M}_{FG}^{(1)} \arrow[r] \arrow[d] & B\mathbb{G}_a \\
\mathrm{Spec}(R) \arrow[uuuuur, dashed, "\text{choice of coordinate}"] \arrow[ur, dashed, "\text{coordinatizability}"] \arrow[r, "\mathbb{G}"] & \mathcal{M}_{FG} \arrow[r, "\omega"] & B\mathbb{G}_m
\end{tikzcd}$$

Given a formal group $\mathbb{G}$, $\mathbb{G}$ is coordinatizable if and only if $\omega(\mathbb{G})$ is trivializable, i.e. iff the composite $\omega \circ \mathbb{G}$ is trivial, i.e. iff $\mathbb{G}$ factors thru the fiber $\mathcal{M}_{FG}^{(1)}$, i.e. iff $\mathbb{G}$ may be equipped with a 1-jet. A choice of coordinate on $\mathbb{G}$ is a lift all the way to $\mathrm{Spec}(L)$ as indicated in the diagram, i.e. to realize $\mathbb{G}$ as a formal group *law* and to fix a map from the Lazard ring classifying it. But since the fibers above the first step of the diagram are all $\mathbb{G}_a$'s, there is no obstruction to lifting each step, as we saw in the proof of Theorem 2.1.64. A lift to $\mathcal{M}_{FG}(n)$ in the above diagram may be thought of as a choice of coordinate of $\mathbb{G}$ through degree n, or as equipping $\mathbb{G}$ with an n-jet.

2.1.5 $(\mathcal{M}_{FG})_{(p)}$ and height

We finish this section by localizing $\mathcal{M}_{FG}$ at a prime and showing that it then admits a filtration by *height*. We fix a prime p for the rest of the section.

Definition 2.1.72. For n a positive integer, let the $[n]$-series of a formal group law F over R be defined inductively by $[1]_F(x) = x$ and

$$[n]_F(x) = F(x, [n-1]_F(x))$$

Let v_n denote the coefficient of x^{p^n} in the power series $[n]_F(x)$ and note that $v_0 = p$. We say F has *height* $\geq n$ if $v_i = 0$ for all $i < n$, and we say F has height exactly n if F has height $\geq n$ and $v_n \in R^\times$.

Example 2.1.73. Let $R = \mathbb{F}_p$, one checks that

$$[p]_{\widehat{\mathbb{G}_a}}(x) = 0$$

and

$$[p]_{\widehat{\mathbb{G}_m}}(x) = x^p$$

Therefore $\widehat{\mathbb{G}_a}$ has height $\geq n$ for every n, i.e. $\widehat{\mathbb{G}_a}$ has height ∞, and $\widehat{\mathbb{G}_m}$ has height 1. It is not difficult to show that height is an invariant of a formal group law, and hence we see that over $\mathbb{F}_p$, $\widehat{\mathbb{G}_a}$ and $\widehat{\mathbb{G}_m}$ are not isomorphic.

Remark 2.1.74. We will not need this notion, but it makes sense to ask for the height of a formal group, i.e. the notion of height makes sense not just in $\mathcal{M}_{FG}(1)$ but also in

$\mathcal{M}_{FG}$. Indeed, a formal group $\mathbb{G}$ still admits an endomorphism $[p] : \mathbb{G} \to \mathbb{G}$ and one defines the height of $\mathbb{G}$ to be $\geq n$ if $[p]$ factors through the Frobenius on $\mathbb{G}$ at least $n-1$ times. See [27, Definition 5.2] for more details.

It turns out that, working p-locally, the coefficients v_n defined above capture all the information about a formal group $\mathbb{G}$ within the stack $(\mathcal{M}_{FG})_{(p)}$, i.e. up to isomorphism and faithfully flat base change. More precisely we have the following:

Proposition 2.1.75. *Let $f : \mathrm{Spec}(\mathbb{Z}_{(p)}[v_1, v_2, \ldots]) \to (\mathcal{M}_{FG})_{(p)}$ be the morphism classified by pushing forward the universal formal group law along the map*

$$L = \mathbb{Z}[x_1, x_2, \ldots] \to \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$$

sending $x_i \mapsto 0$ for $i \neq p^k - 1$ and $x_{p^k-1} \mapsto v_k$. Then f is faithfully flat.

Proof. See Example 7 in [58, Lecture 15]. The claim made therein is that f is flat, but the proof shows it is in fact also faithful. ☐

Remark 2.1.76. One may establish the above proposition using the so-called *Landweber criterion* for flatness, which we discuss in the next section. Note that by Theorem 2.1.44, the pullback diagram

$$\begin{tikzcd}
\mathrm{Spec}(B) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{Z}_{(p)}[v_1, v_2, \ldots]) \arrow[d] \\
\mathrm{Spec}(\mathbb{Z}_{(p)}[v_1, v_2, \ldots]) \arrow[r] & (\mathcal{M}_{FG})_{(p)}
\end{tikzcd}$$

- where $B \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \otimes_L L[b_0^{\pm}, b_1, \ldots] \otimes_L \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ - gives an equivalence of stacks

$$\mathcal{M}_{(\mathbb{Z}_{(p)}[v_1, v_2, \ldots], B)} \to (\mathcal{M}_{FG})_{(p)}$$

The above proposition suggests that the $[p]$-series of a formal group and the notion of height may be capturing a lot of information about $(\mathcal{M}_{FG})_{(p)}$; we therefore consider the filtration induced on $(\mathcal{M}_{FG})_{(p)}$ by height.

Definition 2.1.77. Let $I_n \subset \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ be the ideal $(p, v_1, \ldots, v_{n-1})$. The moduli stack of formal groups of height $\geq n$ is the stack

$$\mathcal{M}_{FG}^{\geq n} := \mathcal{M}_{(\mathbb{Z}_{(p)}[v_1,v_2,\ldots]/I_n, B/I_n)}$$

It is straightforward to check that $\mathcal{M}_{FG}^{\geq n}$ is a closed substack of $(\mathcal{M}_{FG})_{(p)}$, and we define the moduli stack of formal groups of height $\leq n$

$$\mathcal{M}_{FG}^{\leq n} := (\mathcal{M}_{FG})_{(p)} \smallsetminus \mathcal{M}_{FG}^{\geq n+1}$$

Remark 2.1.78. In order to know that $(\mathbb{Z}_{(p)}[v_1, v_2, \ldots]/I_n, B/I_n)$ is indeed a Hopf algebroid, one needs to know that I_n is an invariant ideal, i.e. that $\eta_R(I_n) \subset \eta_L(I_n) \cdot B$. We will return to this notion in the next section.

Example 2.1.79.

1. Note that if F is a formal group law over a p-local ring R, F has height ≤ 0 if and only if R is a $\mathbb{Q}$-algebra. The theory of formal groups over $\mathbb{Q}$-algebras is quite simple: one can show that for every formal group law F over a $\mathbb{Q}$-algebra R, there exists a unique strict isomorphism

$$\log : F \to \widehat{\mathbb{G}_a}$$

called the logarithm of F (see [77, Appendix A2]). Moreover the automorphism group of the formal group law $\widehat{\mathbb{G}_a}$ over a $\mathbb{Q}$-algebra R is isomorphic to $R^\times$, where $r \in R^\times$ corresponds to the power series $f(x) = rx$. The *strict* automorphism group is therefore trivial. This gives the following identifications

   - $\mathcal{M}_{FG} \times \mathrm{Spec}(\mathbb{Q}) \simeq B\mathbb{G}_m \times \mathrm{Spec}(\mathbb{Q})$
   - $\mathcal{M}_{FG}(1) \times \mathrm{Spec}(\mathbb{Q}) \simeq \mathrm{Spec}(\mathbb{Q})$
   - $\mathcal{M}_{FG}^{\leq 0} \simeq \mathcal{M}_{FG} \times \mathrm{Spec}(\mathbb{Q}) \simeq B\mathbb{G}_m \times \mathrm{Spec}(\mathbb{Q})$

2. In [55], Lazard proved that, over an algebraically closed field k, any two formal group laws are isomorphic if and only if they have the same height $0 \leq h \leq \infty$ (see also [58,

Lecture 14]), and over any such k, for all n, there exists a formal group law of height n over k. In particular, fix a formal group law F_n of height n over k, and let G_n be its automorphism group. One has an equivalence

$$\mathcal{M}_{FG}^{=n} \times \mathrm{Spec}(k) \simeq BG_n \times \mathrm{Spec}(k)$$

We now want to give a presentation of the stack $\mathcal{M}_{FG}^{\leq n}$ using our results on locally presentable stacks; this will strengthen our understanding of the filtration of $(\mathcal{M}_{FG})_{(p)}$ by the open substacks $\mathcal{M}_{FG}^{\leq n}$. We need first a criterion of Landweber.

Proposition 2.1.80. *(Landweber's exact functor theorem) The map classifying a formal group law* $\mathrm{Spec}(R) \xrightarrow{F} (\mathcal{M}_{FG})_{(p)}$ *is flat if the sequence* $(p, v_0, v_1, \ldots)$ *is a regular sequence in the ring* R*.*

Proof. The original reference is [54]. We refer the reader to [44, Section 21] for a stacks theoretic proof. □

Proposition 2.1.81. *Let* $f : \mathrm{Spec}(\mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm}]) \to \mathcal{M}_{FG}^{\leq n}$ *be classified by the map* $\mathbb{Z}_{(p)}[v_1, v_2, \ldots] \to \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm}]$*. Letting* $R_n \coloneqq \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm}]$*, we have the following:*

1. *The map* f *is a faithfully flat cover, and it induces an equivalence*

$$\mathcal{M}_{(R_n, R_n \otimes_{L_{(p)}} L_{(p)}[b_0^{\pm}, b_1, \ldots] \otimes_{L_{(p)}} R_n)} \to \mathcal{M}_{FG}^{\leq n}$$

2. *The map* f *factors through* $\mathcal{M}_{FG}(1)$*, and the map*

$$f : \mathrm{Spec}(R_n) \to \mathcal{M}_{FG}^{\leq n}(1)$$

is a faithfully flat cover, where $\mathcal{M}_{FG}^{\leq n}(1)$ *is defined by the pullback*

$$\begin{tikzcd} \mathcal{M}_{FG}^{\leq n}(1) \arrow[r] \arrow[d] & (\mathcal{M}_{FG}(1))_{(p)} \arrow[d] \\ \mathcal{M}_{FG}^{\leq n} \arrow[r] & (\mathcal{M}_{FG})_{(p)} \end{tikzcd}$$

The map f therefore induces an equivalence

$$\mathcal{M}_{(R_n, R_n \otimes_{L_{(p)}} L_{(p)}[b_1,\ldots] \otimes_{L_{(p)}} R_n)} \to \mathcal{M}_{FG}^{\leq n}(1)$$

Proof. For (1), $\mathcal{M}_{FG}^{\leq n}$ is locally presentable by Lemma 2.1.35. By the lemma below, the composition

$$\mathrm{Spec}(R_n) \xrightarrow{f} \mathcal{M}_{FG}^{\leq n} \hookrightarrow (\mathcal{M}_{FG})_{(p)}$$

is flat. f is therefore flat by the lemma below.

To show that f is faithful, note that it suffices to show that for any map $x_A : \mathrm{Spec}(A) \to \mathcal{M}_{FG}^{\leq n}$, the pullback $(x_A)^* f$ is faithfully flat. In general, a flat map of schemes is faithfully flat if and only if it is surjective. A map of schemes $X \to Y$ is surjective if and only if it induces a surjection on functor of points $X(k) \to Y(k)$ for every algebraically closed field k. It follows that an affine flat map of stacks is faithfully flat if and only if it is essentially surjective on k-points for all algebraically closed fields k. By Lazard's theorem in item (2) of Example 2.1.79, for k of characteristic p, $\mathcal{M}_{FG}^{\leq n}(k)$ has exactly n connected components, one for each height ≥ 1, and for k of characteristic zero, $\mathcal{M}_{FG}^{\leq n}(k)$ has one connected component. In characteristic p, for each $1 \leq i \leq n$, the map $R_n \to \mathbb{F}_p$ that sends $v_j \mapsto 0$ for $i \neq i, n$ and $v_i, v_n \mapsto 1$ determines a height i formal group law over $\mathbb{F}_p$, which we then push forward to k to hit the i-th connected component. We argue similarly in the characteristic zero case.

For (2), Here, we have used the fact $\mathrm{Spec}(R_n) \to \mathcal{M}_{FG}^{\leq n}(1)$ is flat by the lemma below, but it is not immediate from (1) that it is faithful because the $\mathbb{G}_m$-torsor $\mathcal{M}_{FG}(1)$ has more points than its quotient $\mathcal{M}_{FG}$. For this, we must use the theory of p-typical formal group laws over a $\mathbb{Z}_{(p)}$-algebra, and we refer the reader to [77, Appendix A2] for more details. Since f hits every point of $\mathcal{M}_{FG}^{\leq n}(1)/\mathbb{G}_m \simeq \mathcal{M}_{FG}^{\leq n}$, it suffices to show that for a given formal group law F of height $i \leq n$ over an algebraically closed field k, f hits every point in the $\mathbb{G}_m$-orbit of F. We again omit the characteristic zero argument and let k have characteristic p. We use the Araki coordinate when speaking of p-typical formal group laws, and we give the argument for when $i < n$; the $i = n$ case is similar.

Let F be the p-typical formal group law over $\mathbb{F}_p$ defined by the map

$$R_n \xrightarrow{F} \mathbb{F}_p$$

sending $v_i, v_n \mapsto 1$ and $v_j \mapsto 0$ for $j \neq i, n$. By Lazard's theorem, if G is a formal group law of height i over a field of characteristic p then there exists some isomorphism $\phi : F \to G$ over an algebraically closed field k. If ϕ is a strict isomorphism, then the isomorphism ϕ lifts to $\mathcal{M}_{FG}(1)$, and f hits the point corresponding to G. If $\phi(x) \equiv ux \mod x^2$ for some $u \in k^\times$, then one has a *strict* isomorphism

$$\psi : ({}^uF) \to G$$

where

$${}^uF(x,y) = uF(u^{-1}x, u^{-1}y)$$

and

$$\psi(x) = \phi(u^{-1}x)$$

To show that f hits the point corresponding to G in $\mathcal{M}_{FG}(1)$, it thus suffices to show that f hits the point corresponding to ${}^uF$ in $\mathcal{M}_{FG}(1)$. One checks that since F is p-typical, and

$$[p]_F(x) = x +_F x^{p^i} +_F x^{p^n}$$

it follows that ${}^uF$ is p-typical with

$$[p]_{{}^uF}(x) = x +_{{}^uF} u^{1-p^i} x^{p^i} +_{{}^uF} u^{1-p^n} x^{p^n}$$

Therefore the map $R_n \to k$ sending $v_i \mapsto u^{1-p^i}$, $v_n \mapsto u^{1-p^n}$, and $v_j \mapsto 0$ for $j \neq i, n$ is sent by f to the point corresponding to G. $\square$

Lemma 2.1.82. *Suppose that the composite* $\mathrm{Spec}(R) \xrightarrow{f} \mathcal{M} \hookrightarrow \mathcal{N}$ *is flat, for $\mathcal{M}$ and $\mathcal{N}$ locally presentable, and $\mathcal{M} \subset \mathcal{N}$ is a (full) open substack. Then f is flat.*

Proof. Suppose we have the following diagram

$$\begin{tikzcd}
\mathrm{Spec}(T) \arrow[r] \arrow[d, "f'"] & \mathrm{Spec}(R) \arrow[d, "f"] \\
\mathrm{Spec}(S) \arrow[r] \arrow[d, "="] & \mathcal{M} \arrow[d, hook] \\
\mathrm{Spec}(S) \arrow[r] & \mathcal{N}
\end{tikzcd}$$

where the upper square is a pullback. It suffices to show that f' is flat. If we knew the bottom square were a pullback, then the big rectangle would be a pullback and then f' would be flat since the righthand vertical composite is flat by assumption. One can check explicitly that the bottom square is a pullback by checking that

$$(\mathrm{Spec}(S) \times_{\mathcal{N}} \mathcal{M})(A)$$

is discrete with objects $\mathrm{Spec}(S)(A)$ for any ring A. □

These open substacks $\mathcal{M}_{FG}^{\leq n}$ are in some sense the only ones one can construct. We will make this precise by first recalling the following result of Landweber.

Proposition 2.1.83. *(Landweber's invariant prime ideal theorem) Let* (A, Γ) *be a Hopf algebroid and* $I \subset A$ *an ideal. We say* I *is invariant if* $\eta_R(I) \subset \eta_L(I) \cdot \Gamma$*. For*

$$(A, \Gamma) = (L_{(p)}, L_{(p)}[b_0^{\pm}, b_1, \ldots])$$

the only invariant prime ideals of $L_{(p)}$ *are the ideals*

$$I_n = (p, v_1, \ldots, v_{n-1})$$

for $0 \leq n \leq \infty$*.*

Proof. See [79, Theorem 3.3.5]. □

We define now the associated space of a stack. It is a useful invariant that encodes theorems such as Lazard's theorem on height and Landweber's invariant prime ideal theorem.

Definition 2.1.84. Let $\mathcal{M}$ be a stack, we define a topological space $|\mathcal{M}|$ - called the *associated space* of the stack $\mathcal{M}$ - as follows. $|X|$ has points consisting of the set of objects in $X(k)$ for fields k modulo the equivalence relation that if $x_i \in X(k_i)$ for $i = 1, 2$, then $x_1 \simeq x_2$ if there is a common field extension K of k_i so that x_1 and x_2 become isomorphic in $X(K)$. The open sets in $|X|$ are of the form $|\mathcal{U}|$ for $\mathcal{U}$ an open substack of X.

Lemma 2.1.85. *Let (A, Γ) be a Hopf algebroid. Every closed substack of $\mathcal{M}_{(A,\Gamma)}$ is of the form $\mathcal{M}_{(A/I,\Gamma/I)}$ for I an invariant ideal.*

Proof. If $\mathcal{Z} \subset \mathcal{M}_{(A,\Gamma)}$ is a closed substack, we may form the pullback square

$$\begin{tikzcd}
\mathrm{Spec}(A/I) \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\
\mathrm{Spec}(A) \arrow[r] & \mathcal{M}_{(A,\Gamma)}
\end{tikzcd}$$

Since $\mathcal{Z}$ is a substack, it follows that I is invariant. $\mathrm{Spec}(A/I) \to \mathcal{Z}$ is a faithfully flat cover, and by Lemma 2.1.35, $\mathcal{Z}$ is locally presentable, so it suffices to observe that in the following diagram

$$\begin{tikzcd}
\mathrm{Spec}(\Gamma/I) \arrow[r] \arrow[d] & \mathrm{Spec}(A/I) \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d] \\
\mathrm{Spec}(A/I) \arrow[r] & \mathcal{Z} \arrow[r] & \mathcal{M}_{(A,\Gamma)}
\end{tikzcd}$$

the right and outer squares are pullbacks, so the left square is a pullback. The result then follows as usual from Theorem 2.1.44. □

Theorem 2.1.86. *There is a homeomorphism from the space $|(\mathcal{M}_{FG})_{(p)}|$ to the space with underlying set $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ and nonempty proper open sets precisely those of the form*

$$U_n = \{i \in \mathbb{Z}_{\geq 0} | i \leq n\}$$

Proof. Lazard's theorem on height immediately implies that the underlying set is $\mathbb{Z}_{\geq 0} \cup \{\infty\}$, and the corresponding map

$$|(\mathcal{M}_{FG})_{(p)}| \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

is continuous because $|\mathcal{M}_{FG}^{\leq n}| = U_n$. Conversely, if $|\mathcal{U}| \subset |(\mathcal{M}_{FG})_{(p)}|$ is a proper open subset, we have a closed substack $\mathcal{Z} := (\mathcal{M}_{FG})_{(p)} \smallsetminus \mathcal{U}$ with $|\mathcal{Z}| = |(\mathcal{M}_{FG})_{(p)}| \smallsetminus |\mathcal{U}|$ nonempty and proper. By Lemma 2.1.85, we have an equivalence

$$\mathcal{Z} \simeq \mathcal{M}_{(L_{(p)}/I, L_{(p)}[b_0^{\pm}, b_1, \ldots]/I)}$$

for I a nonzero, proper invariant ideal. For the rest of the proof, we implicitly work p-locally, in particular writing L to mean $L_{(p)}$.

Because we are working over $\mathcal{M}_{FG} \simeq \mathcal{M}_{FG}(1)/\mathbb{G}_m$, I must in particular be a homogeneous ideal with respect to the grading on the Lazard ring $L \cong \pi_*(MU)$. Indeed, an ideal $I \subset L$ is invariant with respect to the Hopf algebroid $(L, L[b_0^{\pm}, b_1, \ldots])$ if and only if it is homogeneous and invariant with respect to the Hopf algebroid $(L, L[b_1, b_2, \ldots])$.

We may therefore apply the results of [52, Section 2]. In particular, we must have

$$(p) \subset I \subset I_\infty$$

(see [52, pg. 277]). We say $\mathrm{Spec}(L/I)$ has a height n point if there exists a field k and a ring map $L/I \to k$ that carries the universal formal group law over L to a height n formal group law over k. We claim that either

$$|\mathcal{Z}| = |\mathcal{M}_{FG}^{\infty}|$$

or there exists a positive integer n_I such that $\mathrm{Spec}(L/I)$ has a height m point if and only if $m \geq n_I$. This completes the proof as in the latter case,

$$|\mathcal{Z}| = |\mathcal{M}_{FG}^{\geq n_I}|$$

Let n be the largest positive integer such that $I_n \subset I$ (note $n \geq 1$ as $I_1 = (p)$). We construct inductively a sequence of ideals

$$I_n = J_n \subset J_{n+1} \subset \cdots \subset I$$

J_m for $m \geq n$ such that

- J_m is an invariant ideal
- $J_m \subset I_m$
- $\operatorname{Spec}(L/J_m)$ has no points of height $< m$.

We begin for $m = n$ by setting $J_n = I_n$.

Assume then that the ideals J_i for $i \leq m$ have been constructed with the stated properties. If $J_m = I$, then since $J_m \subset I_m$, we have a factorization

$$\begin{tikzcd} \operatorname{Spec}(L/I_m) \arrow[rr] \arrow[dr] & & \mathcal{M}_{FG} \\ & \operatorname{Spec}(L/J_m) \arrow[ur] & \end{tikzcd}$$

and it follows that the points of $\operatorname{Spec}(L/I_m)$ are contained in the points of $\operatorname{Spec}(L/J_m)$. Conversely, since $\operatorname{Spec}(L/J_m)$ has no points of height $< m$, it follows that two sets of points coincide. We then set $n_I := m$ and stop the construction here.

If $J_m \neq I$, then by [53, Lemma 2.10], the comodule I/J_m must have a nonzero primitive y. One has an inclusion of primitives

$$\operatorname{Prim}(I/J_m) \subset \operatorname{Prim}(L/J_m)$$

From the short exact sequence of comodules

$$0 \to I_m/J_m \to L/J_m \to L/I_m \to 0$$

and the fact that

$$\operatorname{Prim}(L/I_m) \cong \mathbb{F}_p[v_m]$$

(see [52, Proposition 2.11]), we have that there exists $r \geq 0$ and $z \in I_m$ such that

$$y = v_m^r + z$$

We define the ideal

$$K_{m+1} := J_m + (y)$$

and the invariant ideal

$$J_{m+1} := \bigcap_{\substack{J \text{ homogeneous} \\ \text{inv. ideal} \\ J \supset K_{m+1}}} J$$

to be the invariant closure of K_{m+1}. The intersection of homogeneous ideals is a homogeneous ideal, and the intersection of invariant ideals for $(L, L[b_1, \ldots])$ is an invariant ideal, as $I \subset L$ is an invariant ideal in this case if and only if it is invariant under the action of the group of power series over $\mathbb{Z}$

$$f(x) = x + b_1 x^2 + b_2 x^3 + \cdots$$

under composition (see [79, Proposition B.5.17]). Since $J_m \subset J_{m+1}$, and $\operatorname{Spec}(L/J_m)$ has no height $< m$ points, $\operatorname{Spec}(L/J_{m+1})$ also has no height $< m$ points. Suppose that $\operatorname{Spec}(L/J_{m+1})$ had a height m point, then the corresponding map

$$f : L \to L/J_{m+1} \to k$$

would satisfy $f(I_m) = 0$ since the formal group law has height m and $f(y) = 0$ since $y \in J_{m+1}$, and therefore

$$f(v_m^r) = 0 \implies f(v_m) = 0$$

as k is a field. This contradicts the fact that f classifies a height m formal group law, and this completes the induction.

Finally if $J_m \neq I$ for all m, then $\operatorname{Spec}(L/I)$ has no height m points for any finite m, and hence we must have in this case

$$|\mathcal{Z}| = |\mathcal{M}_{FG}^{\infty}| = \{\infty\}$$

as $|\mathcal{Z}|$ is nonempty. □

2.2 Chromatic homotopy

In this section, we assume that the reader has a basic familiarity with the homotopy theory of spectra. To fix notions, we work in the symmetric monoidal $(\infty,1)$-category of spectra, **Sp**, as constructed by Lurie [59]. The homotopy category of spectra was first constructed by Boardman in [13]; Lurie's construction lifts Boardman's to quasicategories. Chromatic homotopy theory studies the robust connection between stable homotopy theory and the theory of formal groups. This connection comes from a theorem of Quillen [76], which gives a refinement of complex cobordism homology $MU_*(-)$ to a functor

$$\mathcal{F} : \mathbf{Sp} \to \mathbf{QCoh}(\mathcal{M}_{FG}(1))$$

This recasts classical computations in the Adams-Novikov spectral sequence as sheaf cohomology computations over $\mathcal{M}_{FG}$ and gives a powerful conceptual approach to stable homotopy theory. In particular, lifting the height filtration of $(\mathcal{M}_{FG})_{(p)}$ along the above functor gives a filtration of the category of spectra. The rigidity of $(\mathcal{M}_{FG})_{(p)}$ - as captured by Theorem 2.1.86 - is strongly reflected in the category of finite spectra; this is the subject of the Ravenel conjectures, which we discuss in Section 2.2.3. We begin in Section 2.2.1 by defining the notion of a complex-oriented cohomology theory, and we show that each such theory determines a formal group law. In Section 2.2.2, we define the functor $\mathcal{F}$ and show that it admits a section on a certain locus of Landweber flat sheaves, via the Landweber exact functor theorem.

By a homotopy commutative ring spectrum, we mean a commutative monoid object in the symmetric monoidal category $(\mathrm{Ho}(\mathbf{Sp}), \wedge, \mathbb{S})$.

2.2.1 Complex-oriented cohomology theories

The primary objects of study in chromatic homotopy are *complex-oriented cohomology theories*. We begin with the definition of a complex orientation:

Definition 2.2.1. Let E be a homotopy commutative ring spectrum. A *complex orientation* of E is a class $x \in \tilde{E}^2(\mathbb{CP}^\infty)$ such that the restriction map

$$\tilde{E}^2(\mathbb{CP}^\infty) \to \tilde{E}^2(\mathbb{CP}^1) = \tilde{E}^2(S^2) \cong \pi_0(E)$$

sends x to 1. We say E is *complex-orientable* if there exists a complex orientation of E.

The space $\mathbb{CP}^\infty$ classifies complex line bundles. There is therefore a map

$$\mu : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$$

classifying the line bundle $\pi_1^*\mathcal{L} \otimes \pi_2^*\mathcal{L}$ over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$, where $\pi_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ is the projection map on the i-th factor, and $\mathcal{L}$ is the tautological bundle over $\mathbb{CP}^\infty$. This map participates in the structure of an abelian group object on $\mathbb{CP}^\infty$ in the category of spaces. Therefore, for any homotopy commutative ring spectrum E, one has a commutative group scheme

$$\mathrm{Spec}(E^*(\mathbb{CP}^\infty))$$

Moreover, the map μ restricts to a map $\mathbb{CP}^n \times \mathbb{CP}^m \to \mathbb{CP}^{n+m}$, which gives the ind-system of spaces

$$\{\mathbb{CP}^1 \to \mathbb{CP}^2 \to \mathbb{CP}^3 \to \cdots\}$$

the structure of an abelian group object in the ind category of spaces, and hence one has an abelian group object

$$\mathbb{G}_E = \mathrm{Spf}(E^*(\mathbb{CP}^\infty)) := \{\mathrm{Spec}(E^*(\mathbb{CP}^1)) \to \mathrm{Spec}(E^*(\mathbb{CP}^2)) \to \cdots\}$$

in $\mathbf{FormalSch}_{/\mathrm{Spec}(E_*)}$, provided that E has Kunneth isomorphisms

$$E^*(\mathbb{CP}^n \times \mathbb{CP}^m) \cong E^*(\mathbb{CP}^n) \otimes_{E^*} E^*(\mathbb{CP}^m)$$

for all n, m. It is sufficient for E to admit an isomorphism of formal schemes

$$\mathbb{G}_E \cong \mathrm{Spf}(E^*[[x]])$$

in order for E to possess these Kunneth isomorphisms, and the complex-oriented cohomology theories are precisely those theories E that admit such an isomorphism of formal schemes; that is, those theories E such that $\mathbb{G}_E$ can be represented by a formal group law over E_*.

If, on one hand, one has an isomorphism $\mathbb{G}_E \cong \mathrm{Spf}(E^*[[x]])$, then one has an isomorphism of pro-E_*-algebras $\{E^*(\mathbb{CP}^{n-1})\} \cong \{E^*[x]/x^n\}$. Since

$$\lim{}^1 E^*[x]/x^n = 0$$

and

$$\varprojlim_n E^*[x]/x^n \cong E^*[[x]]$$

it follows from the Milnor sequence that there is an isomorphism $E^*(\mathbb{CP}^\infty) \cong E^*[[x]]$, and the class x is a complex orientation. Conversely, if E is complex-orientable, one has the following.

Proposition 2.2.2. *Suppose E is complex-orientable, then there is an isomorphism of formal schemes*

$$\mathrm{Spf}(E^*\mathbb{CP}^\infty) \cong \mathrm{Spf}(E^*[[x]])$$

over $\mathrm{Spec}(E_*)$*, and in particular an isomorphism*

$$E^*(\mathbb{CP}^\infty) \cong E^*[[x]]$$

of graded E_*-algebras, where x is in cohomological degree 2.

Proof. Fix a complex orientation $x \in \tilde{E}^2(\mathbb{CP}^\infty)$. Since x is a reduced class, and $\mathbb{CP}^{n-1}$ has a cover by n contractible open subsets, there is a well-defined E_*-algebra homomorphism

$$E^*[x]/x^n \to E^*(\mathbb{CP}^{n-1})$$

To show this map is an isomorphism, we use the Atiyah-Hirzebruch spectral sequence, i.e. the spectral sequence

$$E_2 = H^*(\mathbb{CP}^{n-1}; E^*) \implies E^*(\mathbb{CP}^{n-1})$$

arising from the cellular filtration of $\mathbb{CP}^{n-1}$. Give $E^*[x]/x^n$ a descending filtration by putting x in filtration 2; this makes our map $E^*[x]/x^n \to E^*(\mathbb{CP}^{n-1})$ a map of filtered E_*-algebras. The filtration on each is finite, so it suffices to show this map induces an isomorphism on associated graded. The universal coefficient theorem gives an isomorphism $E_2 \cong E^*[x]/x^n$, where x is the restriction of the complex orientation x along the map

$$\tilde{E}^2(\mathbb{CP}^{n-1}) \to \tilde{E}^2(\mathbb{CP}^1) \cong H^2(\mathbb{CP}^{n-1}; E^0)$$

In particular, x is a permanent cycle, and the spectral sequence is one of E_*-algebras, so $E_2 = E_\infty$.

The calculation of $E^*(\mathbb{CP}^\infty)$ then follows from the Milnor sequence; the structure maps in the pro-system $\{E^*[x]/x^n\}$ are all surjective, hence it has vanishing $\lim^1$, and we have an isomorphism

$$E^*(\mathbb{CP}^\infty) \cong \varprojlim E^*(\mathbb{CP}^n) \cong E^*[[x]]$$

$\square$

Remark 2.2.3. Classically, one has an isomorphism of graded rings

$$f : H^*(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[x]$$

where x is the generator in degree 2. In particular, x is a complex orientation of ordinary cohomology, $H\mathbb{Z}$, but the isomorphism f is off by a completion from the isomorphism provided by the previous proposition. Notice that $\mathbb{Z}[x]$ and $\mathbb{Z}[[x]]$ determine the same underlying graded abelian group, and the multiplication of homogeneous elements in these rings agree, so they are isomorphic as graded rings, and there is no contradiction here. They are not isomorphic as rings, however, and there are two issues at play. One is how we choose to determine a ring R from the data of a *graded ring* R_*. If one defines the ring $R = \bigoplus_n R_n$, then, applying this to the graded ring $H^*(\mathbb{CP}^\infty; \mathbb{Z})$, we have the isomorphism f above as rings. However, if one chooses to define $R = \prod_n R_n$, then we have the isomorphism of the previous proposition. In light of the classical isomorphism f, and the fact that $\mathbb{Z}[x]$ is a free module over the coefficient ring $H\mathbb{Z}_* = \mathbb{Z}$ (whereas $\mathbb{Z}[[x]]$ is not a free $\mathbb{Z}$-module), the former choice would seem more sensible.

We choose, however, to always identify $E^*(\mathbb{CP}^\infty)$ as a ring in the latter sense, and this choice accounts for our other issue: $H\mathbb{Z}$ is bounded above. When E is not bounded above (most of the ring spectra we consider are not bounded above), the calculation of the previous proposition gives an isomorphism of graded E_*-modules

$$E^*(\mathbb{CP}^\infty) \cong E^*[[x]]$$

and this is not isomorphic to $E^*[x]$, even as graded E_*-modules. For example in degree 0, $E^0(\mathbb{CP}^\infty)$ consists of power series

$$\sum a_n x^n$$

where $a_n \in E^{-2n} = \pi_{2n}E$, and hence if infinitely many of the groups $\pi_{2n}E$ for $n \geq 0$ are nonvanishing, this is not isomorphic as an abelian group to the degree 0 component of $E^*[x]$.

Lemma 2.2.4. *If E is complex-orientable, and x_1, x_2 are complex orientations of E, each provides an isomorphism*

$$\mathrm{Spf}(E^*\mathbb{CP}^\infty) \cong \mathrm{Spf}(E^*[[x_i]])$$

via Proposition 2.2.2, and hence each provides a formal group law F_i over E_*. There is a canonical strict isomorphism $f : F_1 \to F_2$ of formal group laws over E_*.

Proof. The zig zag

$$\mathrm{Spf}(E^*[[x_2]]) \leftarrow \mathrm{Spf}(E^*\mathbb{CP}^\infty) \to \mathrm{Spf}(E^*[[x_1]])$$

sends x_2 to some power series

$$f(x_1) = x_1 + \sum_{j\geq 1} b_j x_1^{j+1} \in E^*[[x_1]]$$

The coefficient of x_1 is 1 because both complex orientations must restrict to $1 \in \tilde{E}^2(\mathbb{CP}^1) \cong \pi_0 E$. Moreover, the above zig-zag is an isomorphism of formal groups, where the group structure on $\mathrm{Spf}(E^*[[x_i]])$ is given by F_i. This gives a commutative diagram

$$\begin{tikzcd}
E^*[[x_2]] \arrow[r] \arrow[d] & E^*[[x_2]]\widehat{\otimes}_{E^*}E^*[[x_2]] \arrow[d] \\
E^*[[x_1]] \arrow[r] & E^*[[x_1]]\widehat{\otimes}_{E^*}E^*[[x_1]]
\end{tikzcd}$$

The counterclockwise image of x_1 is $f(F_1(x_1 \otimes 1, 1 \otimes x_1))$, and the clockwise image is $F_2(f(x_1) \otimes 1, 1 \otimes f(x_1))$. □

Remark 2.2.5. In other words, for E complex-orientable, a complex orientation gives a choice of coordinate on the underlying formal group $\mathbb{G}_E$. Conversely, any coordinate on $\mathbb{G}_E$ that differs by a strict isomorphism from a coordinate coming from a complex orientation of E determines a complex orientation of E. Said another way, if E is complex-orientable, there is a canonical structure of a formal group equipped with a 1-jet on $\mathrm{Spf}(E^*\mathbb{CP}^\infty)$, and a complex orientation of E is the same thing as a coordinate on $\mathbb{G}_E$ respecting the 1-jet.

We turn now to our most important example of a complex-oriented cohomology theory: complex bordism.

Example 2.2.6. Let MU be the complex bordism spectrum. We construct MU first by defining $MU(n)$ to be the Thom space of the universal bundle $\mathcal{E}_n \to BU(n)$. The inclusion $U(n-1) \to U(n)$ induces a map $BU(n-1) \to BU(n)$, which pulls back $\mathcal{E}_n$ to the bundle $\mathcal{E}_{n-1} \oplus \underline{\mathbb{C}}$. Applying the Thom space construction to this pullback diagram, one has maps

$$\Sigma^2 MU(n-1) \to MU(n)$$

and we define

$$MU = \mathrm{colim}_n\, \Sigma^{-2n} MU(n)$$

MU has a canonical complex orientation as the zero section of $\mathcal{E}_1 \to \mathbb{CP}^\infty$ determines a homotopy equivalence $\mathbb{CP}^\infty \simeq MU(1)$, which we use to define a complex orientation

$$x : \Sigma^{-2}\mathbb{CP}^\infty \simeq \Sigma^{-2}MU(1) \to MU$$

This is, in fact, the *universal* complex orientation. One can show that MU has a canonical ring structure, and for any homotopy commutative ring spectrum E, there is a bijection

$$\{\text{Ring maps } MU \to E\} \cong \{\text{Complex orientations of } E\}$$

given by pushing forward x along a ring map. See [58, Lecture 6] for more details.

Example 2.2.7. Any homotopy commutative ring spectrum E with the property that $E_{2k+1} = 0$ for all $k \in \mathbb{Z}$ - i.e. that E is *even* - is complex orientable. In this case, the groups $H^p(\mathbb{CP}^\infty; E^q)$ vanish unless p and q are both even; the Atiyah-Hirzebruch spectral sequence computing $E^*\mathbb{CP}^\infty$ thus collapses on the E_2 page by a checkerboard phenomenon, and a generator of $H^2(\mathbb{CP}^\infty; E^0)$ lifts to a complex orientation of E.

The strict isomorphisms given by Lemma 2.2.4 are more useful than they may seem at first glance. The construction $\mathbb{G}_E$ brings complex-oriented cohomology theories E into the theory of formal groups, but the isomorphisms of Lemma 2.2.4 introduce automorphism data for formal groups, and they are responsible for the connection to stacks and, in particular, $\mathcal{M}_{FG}$. This part of the story begins with the following.

Proposition 2.2.8. *Let E be complex-orientable. For any complex orientation*

$$x : MU \to E$$

of E, $MU \wedge E$ has two canonical complex orientations given by

$$\eta_L : MU \simeq S^0 \wedge MU \xrightarrow{\eta_{MU} \wedge x} MU \wedge E$$

and

$$\eta_R : MU \simeq MU \wedge S^0 \xrightarrow{id \wedge \eta_E} MU \wedge E$$

Let $f(x) = x + \sum_{j \geq 1} b_j x^{j+1}$ be the strict isomorphism

$$f : \eta_L^* F \to \eta_R^* F$$

furnished by Lemma 2.2.4, where F is the formal group law over MU_* as in Example 2.2.6. Then the map

$$E_*[b_i] \to MU_*E$$

is an isomorphism of graded E_*-algebras, where $|b_i| = 2i$.

Proof. We refer the reader to [77, Lemma 4.1.7] for a proof. See also [58, Lecture 7]. □

Definition 2.2.9. Let E be a homotopy commutative ring spectrum with the property that the map $E_* \to E_*E$ induced by

$$E \simeq S^0 \wedge E \to E \wedge E$$

is flat. We say in this case that E is *Adams flat.*

Lemma 2.2.10. *Let E be an Adams flat ring spectrum. The pair (E_*, E_*E) has a canonical structure of a graded Hopf algebroid, and for any spectrum X, the E_*-module E_*X has a canonical structure of a graded left (E_*, E_*E)-comodule.*

Proof. The maps

$$\eta_L : E \simeq S^0 \wedge E \to E \wedge E$$

and

$$\eta_R : E \simeq E \wedge S^0 \to E \wedge E$$

induce the maps of the same name in the Hopf algebroid. The multiplication map $E \wedge E \to E$ induces the identity map ϵ, the swap map $E \wedge E \xrightarrow{\tau} E \wedge E$ induces the inversion c, and the map

$$E \wedge E \simeq E \wedge S^0 \simeq E \to E \wedge E \wedge E$$

induces the composition map Δ. For Δ, note that since E_*E is flat over E_*, one has a Kunneth isomorphism

$$\pi_*(E \wedge E \wedge E) \cong E_*E \otimes_{E_*} E_*E$$

The comodule structure on E_*X is induced by the map

$$E \wedge X \simeq E \wedge S^0 \wedge X \to E \wedge E \wedge X$$

using flatness again for the isomorphism

$$\pi_*(E \wedge E \wedge X) \cong E_*E \otimes_{E_*} E_*X$$

$\square$

Theorem 2.2.11. *(Quillen) The complex orientation of MU given by Example 2.2.6 - along with the isomorphism of Proposition 2.2.8 - determines an isomorphism of graded Hopf algebroids*

$$(MU_*, MU_*MU) \cong (L, L[b_1, b_2, \ldots])$$

where the latter is as in Theorem 2.1.60. In particular, one has an equivalence of $\mathbb{G}_m$-stacks

$$\mathcal{M}_{(MU_*, MU_*MU)} \simeq \mathcal{M}_{FG}(1)$$

and hence an equivalence of stacks

$$\mathcal{M}_{(MU_*, MU_*MU)}/\mathbb{G}_m \simeq \mathcal{M}_{FG}$$

Proof. We refer the reader to [76]. See also [58, Lecture 10]. $\square$

2.2.2 $\mathcal{F}_X$ and the Landweber exact functor theorem

Quillen's theorem along with Lemma 2.2.10 refines the complex bordism homology theory $MU_*(-)$ to a functor

$$\mathcal{F}_{(-)} : \mathbf{Sp} \to \mathbf{Comod}_{(MU_*, MU_*MU)} \simeq \mathbf{QCoh}(\mathcal{M}_{FG}(1))$$

where the latter equivalence is Theorem 2.1.54. Since the isomorphism of Quillen's theorem respects gradings, this equivalence restricts to one of $\mathbb{G}_m$-equivariant objects, i.e. a graded

(MU_*, MU_*MU)-comodule - such as MU_*X - is the same thing as a $\mathbb{G}_m$-equivariant quasicoherent sheaf on $\mathcal{M}_{FG}(1)$. In this section, we will establish some basic properties of this functor, and we show in the next section that this functor retains a surprising amount of information about stable homotopy theory. Our first result ties this functor to computations in complex bordism:

Proposition 2.2.12. *There is an isomorphism*

$$\mathrm{Ext}^{s,t}_{(MU_*,MU_*MU)}(MU_*, MU_*X) \cong H^s(\mathcal{M}_{FG}(1); \mathcal{F}_X)_t$$

from the E_2-page of the Adams-Novikov SS of X to the cohomology of the sheaf $\mathcal{F}_X$ on the stack $\mathcal{M}_{FG}(1)$. The internal grading t on the latter comes from the fact that $\mathcal{F}_X$ is a $\mathbb{G}_m$-equivariant quasicoherent sheaf on $\mathcal{M}_{FG}(1)$.

Proof. This follows immediately from the equivalence of Theorem 2.1.54. □

It is natural to ask whether a formal group law over a ring R gives rise to a complex-oriented cohomology theory, and, in particular, if the functor $\mathcal{F}_{(-)}$ admits a section. This question was the motivation for Landweber's exact functor theorem 2.1.80, and one has the following.

Lemma 2.2.13. *Suppose that $L \xrightarrow{F} R_*$ is a map of graded rings classifying a formal group law F with the property that, for all primes p, the sequence $(p, v_1, v_2, \ldots)$ is a regular sequence in the ring R_*. Then the functor $h_*(-)$*

$$X \mapsto MU_*(X) \otimes_{MU_*} R_*$$

is represented by a complex-oriented cohomology theory E_R with $\mathbb{G}_{E_R} \cong F$.

Proof. It suffices to show the above functor $h_*(-)$ sends cofiber sequences $X \to Y \to Z$ to exact sequences

$$h_*(X) \to h_*(Y) \to h_*(Z)$$

as then we may apply Brown representability. Such a sequence determines an exact sequence

$$\mathcal{F}_X \to \mathcal{F}_Y \to \mathcal{F}_Z$$

in $\mathbf{QCoh}(\mathcal{M}_{FG}(1))$, and since

$$\mathrm{Spec}(R_*) \xrightarrow{F} \mathcal{M}_{FG}(1)$$

is flat by Proposition 2.1.80, the sequence of R_*-modules

$$F^*\mathcal{F}_X \to F^*\mathcal{F}_Y \to F^*\mathcal{F}_Z$$

is exact, and this is isomorphic to the sequence

$$h_*(X) \to h_*(Y) \to h_*(Z)$$

□

Remark 2.2.14. It is often the case that one has a commutative ring R equipped with a formal group law F to which they would like to apply the lemma, but R has no natural grading for which $L \xrightarrow{F} R$ is a map of graded rings. One fixes this by working in an *even-periodic* setting. Namely, one replaces MU with the ring spectrum

$$MUP = \bigvee_{n \in \mathbb{Z}} \Sigma^{2n}$$

which has the property that $\pi_0 MUP = \pi_* MU$ carries the universal formal group law. For R and F as above such that, for all primes p, the sequence $(p, v_1, v_2, \ldots)$ is a regular sequence in the ring R, the functor

$$X \mapsto (MUP)_0(X) \otimes_{MUP_0} R$$

is representable by a complex-oriented cohomology theory E_R with $\pi_*(E_R) \cong R[u^\pm]$ where u has degree 2. The inclusion $R \to R[u^\pm]$ allows us to push forward F, and we then define a formal group law over $R[u^\pm]$ by

$$({}^{u^{-1}}F)(x, y) = u^{-1}F(ux, uy)$$

and one has an isomorphism $\mathbb{G}_{E_R} \cong {}^{u^{-1}}F$.

Definition 2.2.15. For E a complex-oriented cohomology theory, if $\mathbb{G}_E$ satisfies the hypotheses of the Landweber exact functor theorem, we say E is *Landweber exact*.

Remark 2.2.16. If E is landweber exact, then one has isomorphisms

$$E_*(X) \cong MU_*(X) \otimes_{MU_*} E_*$$

for any spectrum X. One may use this to show that there is a pullback of stacks

$$\begin{tikzcd} \mathrm{Spec}(E_* F) \arrow[r] \arrow[d] & \mathrm{Spec}(E_*) \arrow[d, "\mathbb{G}_E"] \\ \mathrm{Spec}(F_*) \arrow[r, "\mathbb{G}_F"] & \mathcal{M}_{FG}(1) \end{tikzcd}$$

whenever E and F are Landweber exact. Hopkins' cites this fact as one that piqued his interest in bringing stacks into the world of chromatic homotopy. We return to a generalization of this fact in the final chapter, see Proposition 6.1.6.

Example 2.2.17. As discussed in our analysis of $(\mathcal{M}_{FG})_{(p)}$, the formal group laws over the rings $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ and $\mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm}]$ satisfy the hypotheses of the Landweber exact functor theorem. By the lemma above, we therefore have associated complex oriented cohomology theories, which are called BP and $E(n)$, the Brown-Peterson spectrum and the n-th Johnson-Wilson theory.

Example 2.2.18. There is a completed variant of $E(n)$ that is of central importance to chromatic homotopy, known as Morava E-theory. For k a perfect field of characteristic p, there is an (ungraded) ring

$$W(k)[[u_1, \ldots, u_{n-1}]]$$

that carries the universal deformation of a given formal group law $\mathbb{G}$ of height n over k, where $W(k)$ denotes the Witt vectors of k. The sequence $(p, v_1, \ldots)$ is regular in this ring, and hence we have an even-periodic Landweber exact theory E_n such that

$$\pi_*(E_n) = W(k)[[u_1, \ldots, u_{n-1}]][u^{\pm}]$$

This is known as the Morava E-theory associated to $\mathbb{G}$. Goerss, Hopkins, and Miller showed that E_n admits the structure of an E_∞-ring and that the automorphism group of the formal group $\mathbb{G}$ acts on E_n in the category of E_∞-rings (see [22]). This group action gives equivariance a crucial role in chromatic homotopy, and we will return to this in the next chapter.

The Landweber exact functor theorem allows us to lift certain sheaves along the functor $\mathcal{F}_{(-)}$, and we would like to lift certain geometric constructions, such as restriction to an open substack of $\mathcal{M}_{FG}$, along this functor. This brings us to Bousfield localization.

Definition 2.2.19. If E is a spectrum, we let $\mathcal{Z}_E$ denote the category of E-*acyclics*: the full subcategory of $\mathbf{Sp}$ consisting of all Z such that $E \wedge Z$ is contractible. We let $\mathcal{L}_E$ denote the category of E-*locals*: the full subcategory of $\mathbf{Sp}$ consisting of all X such that $\mathrm{Map}_{\mathbf{Sp}}(Z, X) \simeq *$ for all $Z \in \mathcal{Z}_E$. We say $E, F \in \mathbf{Sp}$ are *Bousfield equivalent* (denoted $\langle E \rangle = \langle F \rangle$) if $\mathcal{Z}_E = \mathcal{Z}_F$.

Proposition 2.2.20. *(Bousfield) For any spectrum E there is a cofiber sequence of spectra*

$$Z_E(X) \to X \to L_E(X)$$

such that $Z_E(X) \in \mathcal{Z}_E$, $L_E(X) \in \mathcal{L}_E$, and the cofiber sequence is natural in X and unique up to homotopy with respect to these properties.

Proof. We refer the reader to [15]. See also [58, Lecture 20]. $\square$

We fix a prime p and focus in particular on the Bousfield localization functor $L_{E(n)}(-)$ on the category of p-local spectra. The following proposition makes precise how $L_{E(n)}$ lifts the localization functor on $\mathbf{QCoh}((\mathcal{M}_{FG})_{(p)})$ given by restriction to the open substack $\mathcal{M}_{FG}^{\leq n}$.

Proposition 2.2.21. *Fix a prime p and let $\mathcal{M}_{FG}^{\leq n}(1)$ be defined by the pullback*

$$\begin{tikzcd}
\mathcal{M}_{FG}^{\leq n}(1) \arrow[r, "\iota_n"] \arrow[d] & \mathcal{M}_{FG}(1) \arrow[d] \\
\mathcal{M}_{FG}^{\leq n} \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

We have the following:

1. *$Z \in \mathcal{Z}_E$ if and only if $\iota_n^* \mathcal{F}_Z = 0$*

2. *For any p-local spectrum X, the map $X \to L_{E(n)}X$ induces an isomorphism*

$$\iota_n^* \mathcal{F}_X \to \iota_n^* \mathcal{F}_{L_{E(n)}X}$$

Proof. Claims (1) and (2) are equivalent by the long exact sequence associated to $Z_{E(n)}(X) \to X \to L_{E(n)}(X)$. Proposition 2.1.81 - along with the Landweber exactness of $E(n)$ - implies that one has equivalence of stacks

$$\mathcal{M}_{(E(n)_*, E(n)_*E(n))} \simeq \mathcal{M}_{FG}^{\leq n}(1)$$

$$\mathcal{M}_{(E(n)_*, E(n)_*E(n))}/\mathbb{G}_m \simeq \mathcal{M}_{FG}^{\leq n}$$

In particular, since $\mathrm{Spec}(E(n)_*) \to \mathcal{M}_{FG}(1)$ is a faithfully flat cover, $\iota_n^* \mathcal{F}_Z = 0$ if and only if the pullback $\iota_n^* \mathcal{F}_Z$ to $\mathrm{Spec}(E(n)_*)$ is zero, but by Landweber exactness, this is the $E(n)_*$-module $E(n)_*Z$. □

Remark 2.2.22. The proof of the above proposition used only two facts: $E(n)$ is Landweber exact, and $\mathrm{Spec}(E(n)_*) \to \mathcal{M}_{FG}^{\leq n}$ is a faithfully flat cover. It is not hard to show the same is true for any Morava E-theory of height n, hence we have Bousfield equivalences

$$\langle E(n) \rangle = \langle E_n \rangle$$

2.2.3 The Ravenel conjectures

We have shown already that many of the structural properties of the stack $\mathcal{M}_{FG}$ may be lifted to the category **Sp** along the functor $\mathcal{F}_{(-)}$. For instance, Proposition 2.2.21 shows that restriction to the open substacks $\mathcal{M}_{FG}^{\leq n}$ lifts to the Bousfield localization functor

$$L_{E(n)} : \mathbf{Sp} \to \mathcal{L}_{E(n)}$$

In his landmark paper [78], Ravenel made a series of conjectures about how strongly the rigidity of $\mathcal{M}_{FG}$ is reflected in **Sp**. All but one of these (the telescope conjecture) was proven by Hopkins and his collaborators, and we refer the reader to [79] for an excellent self-contained account of these results. We collect a few of these results in this section and explain how each fits into our stacks picture of chromatic homotopy.

Theorem 2.2.23. *(The smash product theorem, Hopkins-Ravenel) The Bousfield localization functor $L_{E(n)}(-)$ is smashing. That is, for any spectrum X, the map*

$$L_{E(n)}(S^0) \wedge X \to L_{E(n)}(S^0) \wedge L_{E(n)}(X) \to L_{E(n)}(X)$$

is an equivalence.

The smash product theorem tells us that, on the category **Sp**, $L_{E(n)}(-)$ behaves like the Zariski localization

$$\iota_n^*(-) \cong \mathcal{O}_{\mathcal{M}_{FG}^{\leq n}} \otimes_{\mathcal{O}_{\mathcal{M}_{FG}}} (-)$$

as $L_{E(n)}$ is simply given by smashing with $L_{E(n)}(S^0)$.

Theorem 2.2.24. *(The chromatic convergence theorem, Hopkins-Ravenel) Let X be a p-local finite spectrum, then there is an equivalence*

$$X \simeq \varprojlim_n L_{E(n)}(X)$$

The chromatic convergence theorem states that finite X may be recovered from its localizations $L_{E(n)}(X)$. This reflects a geometric phenomenon in $(\mathcal{M}_{FG})_{(p)}$: the filtration

$$\mathcal{M}_{FG}^{\leq 0} \subset \mathcal{M}_{FG}^{\leq 1} \subset \cdots \subset (\mathcal{M}_{FG})_{(p)}$$

is not exhaustive because of the generic point at ∞ given by $\widehat{\mathbb{G}_a}$, but it is exhaustive from the point of view of finitely presented quasicoherent sheaves, as any such sheaf is determined by its restrictions to each $\mathcal{M}_{FG}^{\leq n}$. See [27, Section 8] for more details.

For the next result in this series, we need some basic definitions in tensor-triangular geometry, and we follow closely the paper [4]. In particular, suppose $\mathcal{T}$ is a tensor-triangulated category, such as **Sp** with tensor product given by $\wedge$, the smash product of spectra.

Definition 2.2.25. A (full) subcategory $\mathcal{J} \subset \mathcal{T}$ is said to be *thick* if

1. $\mathcal{J}$ is closed under suspensions, desuspensions, and retracts

2. If $X, Y \in \mathcal{J}$, then for any map $f : X \to Y$, the cofiber of f is also in $\mathcal{J}$

We say that $\mathcal{J}$ is a *thick tensor ideal* if, in addition, $X \in \mathcal{J}$ and $Y \in \mathcal{T}$ implies that $X \otimes Y \in \mathcal{J}$. Finally $\mathcal{J}$ is said to be *prime* if

$$X \otimes Y \in \mathcal{J} \implies X \in \mathcal{J} \text{ or } Y \in \mathcal{J}$$

When $\mathcal{T}$ is an essentially small category, such as $\mathbf{Sp}^{\omega}$ (the category of finite spectra), we have the following powerful invariant:

Definition 2.2.26. (Balmer) For an essentially small triangulated category $\mathcal{T}$, we define a topological space $\mathrm{Spc}(\mathcal{T})$ called the *Balmer spectrum* of $\mathcal{T}$ as follows:

- As a set $\mathrm{Spc}(\mathcal{T}) = \{\mathcal{J} : \mathcal{J} \text{ is a prime thick tensor ideal of } \mathcal{T}\}$.

- For a family of objects $\mathcal{S} \subset \mathcal{T}$, we define the set

$$Z(\mathcal{S}) \coloneqq \{\mathcal{J} \in \mathrm{Spc}(\mathcal{T}) : S \cap \mathcal{J} = \varnothing\}$$

  The sets $Z(\mathcal{S})$ define the closed subsets of a topology on $\mathrm{Spc}(\mathcal{T})$.

Theorem 2.2.27. *(The thick subcategory theorem, Hopkins-Smith) As a set,*

$$\mathrm{Spc}(\mathbf{Sp}^{\omega}_{(p)}) = \{\mathcal{C}_{\geq n}\}_{n \geq 0}$$

where $\mathcal{C}_{\geq n}$ is the thick subcategory of finite $E(n-1)$-acyclics, i.e.

$$\mathcal{C}_{\geq n} = \mathcal{Z}_{E(n-1)} \cap \mathbf{Sp}^{\omega}_{(p)}$$

The map

$$\mathrm{Spc}(\mathbf{Sp}^{\omega}_{(p)}) \to |(\mathcal{M}_{FG})_{(p)}| \cong \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

sending $\mathcal{C}_{\geq n}$ to n is a homeomorphism onto the subset $\mathbb{Z}_{\geq 0}$.

The thick subcategory theorem is a spectacular lift of the homeomorphism of Theorem 2.1.86. It allows us to treat the category of **Sp** much in the way we treat a scheme, and in particular it allows one to reduce claims about finite spectra to a single generic spectrum.

Theorem 2.2.28. *(The nilpotence theorem, Devinatz-Hopkins-Smith) Let R be a ring spectrum. If $x \in \pi_* R$ is in the kernel of the map*

$$\pi_*(R) \cong \pi_*(S^0 \wedge R) \to \pi_*(MU \wedge R) = MU_* R$$

then x is nilpotent in $\pi_* R$.

The nilpotence theorem is the central result used to establish the Ravenel conjectures. The main ingredient of the proof of the nilpotence theorem is a filtration of MU introduced by Ravenel called the $X(n)$'s. We return to these in the final chapter.

Chapter 3

REAL-ORIENTED HOMOTOPY AND THE SLICE FILTRATION

In this chapter, we review some of the basics of equivariant stable homotopy and Real-oriented homotopy, collecting what we need for our results in the remaining chapters. In Section 3.1 we discuss basics regarding stabilization and genuine G-spectra. In Section 3.2, we discuss Real orientations and show how many of the features of chromatic homotopy and complex orientations are available in this context. In Section 3.3, we introduce the slice filtration of G-spectra and discuss the HHR slice theorem. Finally in Section 3.4, we discuss the Segal conjecture for $G = C_p$ and show that it can be recast as a completion statement on the level of C_p-spectra. In all that follows, G is a finite group.

3.1 G-spaces and G-spectra

In this section, we define the homotopy theories of G-spaces and G-spectra, and discuss various properties. We begin in Section 3.1.1 by discussing stabilization in a G-equivariant context, defining the category of genuine G-spectra. In Section 3.1.2, we discuss various change of group functors in equivariant homotopy and give a description of C_p-spectra in terms of pullback squares.

3.1.1 G-spaces, Mackey functors, and stabilization

Definition 3.1.1. A *G-space* is a topological space X equipped with a continuous G-action $G \times X \to X$. A *map of G-spaces* is a G-equivariant continuous map. A *pointed G-space* is a G-space X with a chosen point $x \in X^G$, and a *map of pointed G-spaces* is a map of G-spaces that is pointed. A *weak equivalence* of pointed G-spaces is a map $f : X \to Y$ of G-spaces such that $f^H : X^H \to Y^H$ is a weak equivalence of pointed spaces for all subgroups $H \subset G$.

There is a natural model category structure on $\mathbf{Top}_*^G$, the category of G-spaces in which the weak equivalences are exactly the ones in the above definition (see [29]), and we denote by the same name $\mathbf{Top}_*^G$ the associated ∞-category given by taking the homotopy coherent nerve of the category of bifibrant objects therein. We define equivariant homotopy groups in the expected way; for X a G-space,

$$\pi_i^G(X) := [S^i, X]^G$$

where $[-,-]^G$ denotes Hom in the homotopy category of $\mathbf{Top}_*^G$, and S^i denotes the i sphere with the trivial G-action. π_i is a group for $i \geq 1$ and an abelian group for $i \geq 2$. Since S^i has the trivial action, any map $S^i \to X$ must land in X^G, and we have

$$\pi_i^G(X) \cong \pi_i(X^G)$$

There is, however, more structure available on the groups $\pi_i^H(X) := \pi_i(X^H)$. In particular, for any subgroup $H \subset G$, the fixed point-set X^H has a residual action of the Weyl group $W_G(H) = N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G. For any $K \subset H \subset G$, there is also an inclusion $X^H \to X^K$. For $i \geq 2$, this gives the collection of abelian groups $\{\pi_i^H(X)\}_{H \subset G}$ the structure of a *coefficient system*:

Definition 3.1.2. Let $\mathcal{O}_G$ denote the *orbit category* of G: the full subcategory of the category of G-sets and equivariant maps spanned by the orbits G/H for H a subgroup of G. A G-*coefficient system* is a functor $\mathcal{O}_G^{op} \to \mathbf{Ab}$.

The coefficient system point-of-view actually gives rise to a useful model of the homotopy theory of G-spaces, according to the following theorem of Elmendorf.

Proposition 3.1.3. *(Elmendorf) There is an equivalence of ∞-categories*

$$\mathbf{Top}_*^G \simeq \mathrm{Fun}(\mathcal{O}_G^{op}, \mathbf{Top}_*)$$

The passage from G-spaces to G-spectra is substantially more subtle than in the nonequivariant context. The *correct* notion of G-spectrum - i.e. a *genuine* G-spectrum - ought to give a representing object for a classical G-equivariant cohomology theory (see [20, Definition 3.3.3]). A G-equivariant cohomology theory E has groups $E^V(X)$ graded on $V \in RO(G)$ - the real representation ring - for any G-space X, along with (natural) twisted suspension isomorphisms

$$\sigma_{V,W} : E^V(X) \xrightarrow{\cong} E^{V+W}(S^W \wedge X)$$

In addition the G-coefficient system

$$G/H \mapsto E^V(G/H \times X)$$

possesses additional *transfer* maps, i.e. it extends to a G-Mackey functor, which we define below (see also [39, Section 3.1]).

Definition 3.1.4. A *G-Mackey functor* consists of a pair $\underline{M} = (\underline{M}_*, \underline{M}^*)$ of functors from the category of finite G-sets to the category of abelian groups. The two functors have the same object function (denoted $\underline{M}$) and take disjoint unions to direct sums. The functor M_* is covariant, while M^* is contravariant, and together they take a pullback diagram of finite G-sets

$$\begin{tikzcd} S \arrow[r, "\delta"] \arrow[d, "\gamma"'] & A \arrow[d, "\alpha"] \\ T \arrow[r, "\beta"] & B \end{tikzcd}$$

to a commutative square

$$\begin{tikzcd} \underline{M}(S) \arrow[r, "\underline{M}_*(\delta)"] & \underline{M}(A) \\ \underline{M}(T) \arrow[u, "\underline{M}^*(\gamma)"] \arrow[r, "\underline{M}_*(\beta)"] & \underline{M}(B) \arrow[u, "\underline{M}^*(\alpha)"'] \end{tikzcd}$$

The contravariant maps $M^*(\alpha)$ are called the *restriction maps*, and the covariant maps $M_*(\beta)$ the *transfer maps.*

Example 3.1.5. 1. We let $\underline{\mathbb{Z}}$ be the Mackey functor given by $\underline{\mathbb{Z}}(G/H) = \mathbb{Z}$ for all H, the restriction maps $\underline{\mathbb{Z}}(G/H) \to \underline{\mathbb{Z}}(G/K)$ are the identity, and the transfer maps $\underline{\mathbb{Z}}(G/K) \to \underline{\mathbb{Z}}(G/H)$ for $K \subset H$ are multiplication by the index $[H : K]$. All the Weyl group actions are trivial.

2. We let $\underline{A}$ be the Burnside Mackey functor given by $\underline{A}(G/H) = A(H)$, where $A(H)$ is the Burnside ring of H. This is the set of isomorphism classes of finite G-sets, where addition is given by disjoint union, and multiplication is given by Cartesian product. The restriction maps are given by restriction of G-sets, the transfer maps are given by induction of G-sets, and the Weyl actions are given as follows: for X a finite H-set and $g \in W_G(H)$, one sets gX to be the H-set X with action $h \cdot x = ghg^{-1}x$.

Mackey functors may also be defined as additive functors from B_G, the Burnside category of G, to **Ab**; we refer the reader to [39, Section 3.1] for more details. The following result implies that the condition of having twisted suspension isomorphisms and the condition of extending to a Mackey functor are actually equivalent, and we comment below on why this is the case in Remark 3.1.7.

Proposition 3.1.6. *A $\mathbb{Z}$-graded cohomology theory on* $\mathbf{Top}^G$ *with coefficients in a coefficient system M extends to an $RO(G)$-graded cohomology theory if and only if M extends to a Mackey functor.*

Proof. See [20, Corollary 3.4.5]. □

Remark 3.1.7. The condition of G-equivariant cohomology theories extending to an $RO(G)$-graded cohomology theory - i.e. possessing twisted suspension isomorphisms - is equivalent to the representation spheres being invertible in the category of G-spectra. This condition gives rise to a Mackey functor because one then has a stable transfer map $\Sigma^\infty_+ G/H \to \Sigma^\infty_+ G/K$ for $K \subset H$ via the Pontryagin-Thom construction. For example, let $G = H = C_2$ and $K = \{e\}$, then the map in question would be a stable map

$$S^0 \to C_{2+}$$

(Notice there is no such nontrivial unstable map that is C_2-equivariant). In the presence of a suspension isomorphism for σ, the sign representation of C_2, it would suffice to suspend by S^σ and thus construct a map

$$S^\sigma \to C_{2+} \wedge S^\sigma \cong C_{2+} \wedge S^1$$

using the equivariant homeomorphism $C_{2+} \wedge S^1 \to C_{2+} \wedge S^\sigma$ that sends

$$(g, x) \mapsto (g, gx)$$

To do this, embed the C_2-set C_2 into σ as the subset $\{\pm 1\}$. Taking small neighborhoods around 1 and -1 gives an equivariant open embedding

$$C_2 \times D^1 \hookrightarrow \sigma$$

The Pontryagin-Thom construction is the observation that one-point compactification on locally compact spaces is a contravariant functor with respect to open inclusions, and this gives our map

$$S^\sigma \to C_{2+} \wedge S^1$$

With this motivation in mind, we thus have two ways of passing from G-spaces to G-spectra, by inverting the representation spheres of G or by extending to Mackey functors; the former approach is taken by Mandell and May in their stable model structure on orthogonal G-spectra [63]. The latter approach is taken by Guillou-May [30] and Barwick in [10]. Barwick constructs a quasicategory of spectral mackey functors that is equivalent to the homotopy coherent nerve of bifibrant objects in the Mandell-May category. We denote either of these equivalent ∞-categories by $\mathbf{Sp}^G$, the category of genuine G-spectra.

Example 3.1.8. Any Mackey functor $\underline{M}$ determines an Eilenberg-Maclane G-spectrum $H\underline{M}$ with the property that

$$\underline{\pi}_i(H\underline{M}) = \begin{cases} \underline{M} & i = 0 \\ 0 & \text{else} \end{cases}$$

by constructing the spectral Mackey functor

$$(H\underline{M})(G/H) = H(\underline{M}(G/H))$$

The homotopy Mackey functors $\underline{\pi}_i(-)$ may be defined more generally even for $V \in RO(G)$ as follows:

Definition 3.1.9. For X a G-spectrum, we define the Mackey functor $\underline{\pi}_V(X)$ by

$$\underline{\pi}_V(X)(G/H) = [G/H_+ \wedge S^V, X]^G$$

where $[-,-]^G$ denotes Hom in the homotopy category of $\mathbf{Sp}^G$.

Remark 3.1.10. The functoriality of the above construction comes from the following result of Segal: the functor $\mathcal{B}_G \to \mathbf{Sp}^G$ sending

$$G/H \mapsto \Sigma_+^\infty G/H$$

gives an equivalence of categories from the Burnside category $\mathcal{B}_G$ onto the full subcategory of $\mathbf{Sp}^G$ spanned by the suspension spectra of finite G-sets. Note in particular, this gives an isomorphism

$$\underline{\pi}_0(S^0) \cong \underline{A}$$

where $\underline{A}$ is the Burnside Mackey functor for G as in Example 3.1.5.

Remark 3.1.11. The category $\mathbf{Sp}^G$ of genuine G-spectra is not the only reasonable homotopy theory of G-spectra. One could, for example, take the ∞-category $\mathbf{Fun}(BG, \mathbf{Sp})$ of G-objects in $\mathbf{Sp}$, or by analogy with Elmendorf's theorem, the ∞-category $\mathbf{Fun}(\mathcal{O}_G^{op}, \mathbf{Sp})$ of G-coefficient systems of spaces. In the former case, one has the category of so-called *Borel* G-spectra, and in the latter case, one has so-called *naive* G-spectra. We will never make use of naive G-spectra, but we will frequently use Borel G-spectra for the following reason: the forgetful functor

$$\mathbf{Sp}^G \xrightarrow{\mathrm{ev}_{G/e}} \mathbf{Fun}(BG, \mathbf{Sp})$$

admits fully faithful left and right adjoints, denoted $EG_+ \wedge (-)$ and $F(EG_+, -)$ respectively. The notation is due to the fact that $EG_+ \wedge (-)$ is an equivalence onto the full subcategory of $\mathbf{Sp}^G$ given by those X such that

$$EG_+ \wedge X \to X$$

is an equivalence, where EG is a free contractible G-space, and $F(EG_+, -)$ is an equivalence onto the full subcategory of $\mathbf{Sp}^G$ given by those X such that

$$X \to F(EG_+, X)$$

is an equivalence. The former subcategory is called the category of *free* G-spectra, and the latter is called the category of *cofree* G-spectra. Cofree G-spectra will play an important role in many of our results.

Example 3.1.12. Recall from Example 2.2.18 that for any formal group law $\mathbb{G}$ of height n over a perfect field k, there is an associated Landweber exact E_∞-ring spectrum E_n, with an action of the automorphism group of $\mathbb{G}$ by E_∞-ring maps, by the Goerss-Hopkins-Miller theorem. For any finite subgroup G of the automorphism group of $\mathbb{G}$, we therefore have a lift

$$E_n \in \mathbf{Fun}(BG, \mathbf{Sp})$$

We often regard E_n as a cofree genuine G-spectrum by applying the right adjoint $F(EG_+, -)$ of the previous remark.

3.1.2 Change of group functors and the Tate square

We finish this section by describing various change of group functors appearing in equivariant stable homotopy, focusing on the geometric fixed point functors in particular to show that when $G = C_p$, it is possible to present the ∞-category $\mathbf{Sp}^G$ as a homotopy pullback via the Tate square. We refer the reader to [39] for more details on the construction of the various change of group functors given below. See also [7, Section 2] for a nice account of many of these functors.

Definition 3.1.13. Let $H \subset G$ be a subgroup. One has the following change of group functors

1. The *restriction* functor $i_H^G : \mathbf{Sp}^G \to \mathbf{Sp}^H$
2. The *induction* functor $G_+ \wedge_H (-) : \mathbf{Sp}^H \to \mathbf{Sp}^G$, the left adjoint to i_H^G.
3. For any group homomorphism $f : G \to G'$, we have the *pullback* functor

$$f^* : \mathbf{Sp}^{G'} \to \mathbf{Sp}^G$$

4. The *inflation* functor $i_* : \mathbf{Sp} \to \mathbf{Sp}^G$ that sends a spectrum to a G-spectrum with trivial action.
5. The *genuine fixed points* functor $(-)^H : \mathbf{Sp}^G \to \mathbf{Sp}$ is given by evaluating a spectral Mackey functor $\underline{X}$ at the G-set G/H. Since $\underline{X}(G/H)$ has a residual action of $W_G(H)$ one may regard this functor as a functor

$$\mathbf{Sp}^G \to \mathbf{Sp}^{W_G(H)}$$

One has by definition that $\pi_i^H(X) \cong \pi_i(X^H)$ and thus the genuine fixed point functors $(-)^H$ are jointly conservative as H ranges through all subgroups of G. The functor $(-)^G$ is right adjoint to the inflation functor, and $(-)^G$ is also a left adjoint. Inflation has no left adjoint.

6. The *geometric fixed points* functor $\Phi^H : \mathbf{Sp}^G \to \mathbf{Sp}$ is the functor sending $X \in \mathbf{Sp}^G$

$$X \mapsto (\tilde{E}\mathcal{F}_{<H} \wedge X)^H$$

where $E\mathcal{F}_{<H+}$ is a universal space for the family of subgroups of G that are subconjugate to a proper subgroup of H, and $\tilde{E}\mathcal{F}_{<H} \wedge X$ is defined by the cofiber sequence

$$E\mathcal{F}_{<H+} \to S^0 \to \tilde{E}\mathcal{F}_{<H}$$

(for more on families and universal spaces, see Section 4.2.1 below). Since $(\tilde{E}\mathcal{F}_{<H} \wedge X)^H$ has a residual action of $W_G(H)$ one may regard this functor as a functor

$$\mathbf{Sp}^G \to \mathbf{Sp}^{W_G(H)}$$

The functors $\Phi^H(-)$ are jointly conservative as H ranges through all subgroups of G. The functor $\Phi^G(-)$ may be described from the perspective of spectral mackey functors: let $\mathcal{C} \subset \mathbf{Sp}^G$ be the full subcategory spanned by those spectral Mackey functors $\underline{X}$ with the property that $\underline{X}(G/H) \simeq *$ for all proper subgroups H. The inclusion of $\mathcal{C}$ admits a left adjoint, and it is not hard to see that $\mathcal{C} \simeq \mathbf{Sp}$. Under this identification, the left adjoint is the functor $\Phi^G(-)$.

7. For $X \in \mathbf{Sp}^G$, we define the *homotopy fixed point spectrum* X^{hH} as

$$(F(EG_+, X))^H$$

As before, one may regard this as a genuine $W_G(H)$-spectrum.

8. For $X \in \mathbf{Sp}^G$, we define the *Tate spectrum* X^{tH} as

$$(\tilde{E}G \wedge F(EG_+, X))^H$$

As before, one may regard this as a genuine $W_G(H)$-spectrum.

9. The *norm functor* $N_H^G : \mathbf{Sp}^H \to \mathbf{Sp}^G$ is a tensor-induction functor. It is symmetric monoidal, it commutes with sifted (and therefore filtered) colimits, and thus it is characterized by its effect on suspension spectra, where we have

$$N_H^G(\Sigma^\infty_+ X) \simeq \Sigma^\infty_+ F_H(G, X)$$

where $F_H(G,-) : \mathbf{Top}^H \to \mathbf{Top}^G$ is the right adjoint to the restriction functor. Note, in particular, if $V \in RO(G)$, one has

$$N_H^G(S^V) \simeq S^{\mathrm{Ind}_H^G(V)}$$

Remark 3.1.14. We pause to say more about the norm in two simple cases. If $X \in \mathbf{Sp}$, then $N_e^{C_2} X$ is a C_2-spectrum whose underlying Borel C_2-spectrum is $X \wedge X$ with the swap action. If $X \in \mathbf{Sp}^{C_2}$, then $N_{C_2}^{C_4} X$ is a C_4-spectrum whose underlying Borel C_4-spectrum is given by $X \wedge X$ with action given heuristically by $\gamma(a \wedge b) = \overline{b} \wedge a$, where γ is a generator of C_4, and $\overline{(-)}$ is the C_2-action on X. This describes the Borel equivariant spectra, and it is more difficult to describe the genuine equivariant spectra, but for $N_e^{C_p}$, this can be done rather explicitly using the Tate diagonal: we refer the reader to [74] for more details.

Definition 3.1.15. Let $X \in \mathbf{Sp}^G$.

1. The cofiber sequence

$$EG_+ \wedge X \to X \to \tilde{E}G \wedge X$$

is called the *isotropy separation sequence* of X.

2. The commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \tilde{E}G \wedge X \arrow[d] \\ F(EG_+, X) \arrow[r] & \tilde{E}G \wedge F(EG_+, X) \end{tikzcd}$$

is called the Tate square of X

Lemma 3.1.16. *For any $X \in \mathbf{Sp}^G$, the Tate square of X is a homotopy pullback in $\mathbf{Sp}^G$.*

Proof. We will see in the next chapter that the functors $F(EG_+, -)$ and $\tilde{E}G \wedge (-)$ may be described as the Bousfield localization functors $L_{G_+}(-)$ and $L_{\tilde{E}G}(-)$. The result then follows from a general argument for Bousfield localizations, which one may find in Bauer's lecture in [22]. One needs only to observe in this case that

$$L_{G_+}(L_{\tilde{E}G}(-)) \simeq *$$

because

$$F(EG_+, \tilde{E}G \wedge X) \simeq F(\tilde{E}G \wedge EG_+ \tilde{E}G, X) \simeq *$$

as $\tilde{E}G \wedge EG_+ \simeq *$. $\square$

Example 3.1.17. Let Y be a genuine C_p-spectrum. Applying $(-)^{C_p}$ to the Tate square for Y, one has a homotopy pullback

$$\begin{tikzcd} Y^{C_p} \arrow[r] \arrow[d] & \Phi^{C_p}(Y) \arrow[d] \\ Y^{hC_p} \arrow[r] & Y^{tC_p} \end{tikzcd}$$

When $X \in \mathbf{Sp}$ and $Y = N_e^{C_p} X$, one has a pullback square

$$\begin{tikzcd} (N_e^{C_p} X)^{C_p} \arrow[r] \arrow[d] & X \arrow[d] \\ (N_e^{C_p} X)^{hC_p} \arrow[r] & (N_e^{C_p} X)^{tC_p} \end{tikzcd}$$

from the identification $\Phi^{C_p} \circ N_e^{C_p} \simeq \mathrm{id}$. The map $X \to (N_e^{C_p} X)^{tC_p}$ is called the Tate diagonal of X.

Remark 3.1.18. The C_p-Tate squares of the lemma above in fact characterize the homotopy theory of genuine C_p-spectra, in the sense that there is a homotopy pullback of ∞-categories

$$\begin{tikzcd} \mathbf{Sp}^{C_p} \arrow[r, "\phi"] \arrow[d, "i_e^{C_p}"'] & \mathbf{Fun}(\Delta^1, \mathbf{Sp}) \arrow[d, "\mathrm{ev}_1"] \\ \mathbf{Fun}(BC_p, \mathbf{Sp}) \arrow[r, "(-)^{tC_p}"] & \mathbf{Sp} \end{tikzcd}$$

where $i_e^{C_p}$ sends a genuine C_p-spectrum X to its underlying Borel C_p-spectrum, and ϕ sends X to the morphism of spectra

$$\Phi^{C_p} X \to X^{tC_p}$$

In particular, the data of a genuine C_p-spectrum is the data of a Borel C_p-spectrum X, a spectrum Y, and a map of spectra

$$Y \to X^{tC_p}$$

See [26] for more details.

3.2 Real orientations and Real bordism theory

Many of the standard complex-oriented cohomology theories - e.g. MU, BP, $E(n)$, $K(n)$ - admit C_2-actions via complex conjugation, resulting in lifts of these spectra to genuine C_2-spectra. Moreover, $\mathbb{CP}^\infty$ - with its complex conjugation action - is an abelian group object equivariantly, i.e. in $\mathbf{Top}_*^{C_2}$. Many of the same conditions are in place to use these objects to have a C_2-equivariant version of chromatic homotopy, which we call Real-oriented homotopy theory. This theory was set in motion by Hu and Kriz in [48], and we also recommend [40] as an excellent source. We collect what we need from the theory in this section.

Remark 3.2.1. For a G-spectrum, we use the notation

$$\underline{\pi}_\bigstar(X) = \bigoplus_{V \in RO(G)} \underline{\pi}_V(X)$$

and we often use the notation $\pi_\bigstar^G X$ and $\pi_V^G X$ or just $\pi_\bigstar X$ and $\pi_V X$ as shorthand for $\underline{\pi}_\bigstar(X)(G/G)$ and $\underline{\pi}_V(X)(G/G)$. Similarly for homology and cohomology groups $E_\bigstar X$ and $E^\bigstar X$.

Note that when $G = C_2$, the real representation ring $RO(C_2)$ is free abelian on the representations 1 and σ, the trivial and sign characters of C_2. Note also that, as a C_2-representation via complex conjugation, $\mathbb{C} \cong 1 + \sigma =: \rho$.

Definition 3.2.2. Let E be a homotopy commutative ring C_2-spectrum. A *Real orientation* of E is a class $x \in \tilde{E}^\rho(\mathbb{CP}^\infty)$ such that the restriction

$$\tilde{E}^\rho(\mathbb{CP}^\infty) \to \tilde{E}^\rho(\mathbb{CP}^1) = \tilde{E}^\rho(S^\rho) \cong \pi_0 E$$

sends x to 1. We say E is Real-orientable if there exists a Real orientation of E.

All of the structure on the ind-space

$$\{\mathbb{CP}^1 \to \mathbb{CP}^2 \to \mathbb{CP}^3 \to \cdots\}$$

we used to define the formal group $\mathbb{G}_E$ for E complex-orientation is C_2-equivariant with respect to complex conjugation on the spaces $\mathbb{CP}^n$. It follows that the ind-system $\{\mathrm{Spec}(E^\bigstar \mathbb{CP}^n)\}$

forms some kind of formal group object; it remains to show that it can be identified with some nice smooth formal group scheme when E is Real-orientable. In fact, the situation is exactly as in the nonequivariant case, but we need a modified version of the Atiyah-Hirzebruch spectral sequence.

Definition 3.2.3. A *Real CW-complex* is a C_2-space X with a filtration

$$X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \cdots \subset X$$

such that

1. $X \cong \operatorname{colim} X^{(n)}$

2. There is a pushout square of C_2-spaces

$$\begin{array}{ccc} \coprod_{I_k} S^{2k-1} & \longrightarrow & X^{(k-1)} \\ \downarrow & & \downarrow \\ \coprod_{I_k} D^{2k} & \longrightarrow & X^{(k)} \end{array}$$

   where I is some indexing set, and S^{2k-1} and D^{2k} are C_2-spaces via complex conjugation coming from their embedding into $\mathbb{C}^k$.

Example 3.2.4. The C_2-spaces $\mathbb{CP}^n$, $\mathbb{CP}^\infty$, $BU(n)$, and $MU(n)$ are all Real CW complexes via their Schubert cell structures.

If E is any C_2-spectrum, and X is a Real CW complex, one gets an Atiyah-Hirzebruch type spectral sequence associated to the Real filtration of X with signature

$$E_1^{p,V} = E^{p\rho_2+V}(X^{(p)}, X^{(p-1)}) \implies E^{p\rho_2+V}(X)$$

One has isomorphisms

$$E^{p\rho_2+V}(X^{(p)}, X^{(p-1)}) \cong \bigoplus_{I_p} E^V(*)$$

On the underlying space, i_e^*X, the Real CW structure forgets to an ordinary CW structure consisting of only even cells. The d^1-differentials in the above spectral sequence correspond to the ordinary cellular boundary maps for i_e^*X, and we find that

$$E_2^{p,V} \cong H^{2p}(i_e^*X; E^V(*)) \implies E^{p\rho_2+V}(X)$$

(see also [48, 2.24]). Using this spectral sequence and the same arguments used in Proposition 2.2.2, we have the following.

Proposition 3.2.5. *A Real orientation x of E determines an isomorphism of $RO(C_2)$-graded rings*

$$E^{\bigstar}(\mathbb{CP}^\infty) \cong E^{\bigstar}[[x]]$$

In particular, x determines a formal group law over the ring

$$\bigoplus_* E_{*\rho} \subset E_{\bigstar}$$

Example 3.2.6. (Real bordism theory) The complex-oriented cohomology theory MU admits a complex conjugation action by C_2. In particular one has the C_2-spaces $MU_{\mathbb{R}}(n)$ - given by the space $MU(n)$ with its complex conjugation action - and the structure maps

$$\Sigma^2 MU(n-1) \to MU(n)$$

lift to C_2-equivariant maps

$$\Sigma^\rho MU_{\mathbb{R}}(n-1) \to MU_{\mathbb{R}}(n)$$

and we define

$$MU_{\mathbb{R}} := \operatorname{colim}_n \Sigma^{-n\rho} MU_{\mathbb{R}}(n)$$

As before the zero section of $\mathcal{E}_1 \to \mathbb{CP}^\infty$ is C_2-equivariant and defines a Real orientation of $MU_{\mathbb{R}}$, which is the universal Real orientation, so that

$$\{C_2\text{-ring maps } MU_{\mathbb{R}} \to E\} \cong \{\text{Real orientations of } E\}$$

Remark 3.2.7. There is a somewhat surprising consequence of this bijection. One can show - just as we did nonequivariantly - that the set of Real orientations of E is in bijection with the set of coordinates on the formal group $\mathbb{G}_E$ over $\mathrm{Spec}(E_{*\rho})$ that respect the 1-jet. We will see that the restriction map

$$MU_{*\rho} \to MU_{2*} \cong L$$

is an isomorphism, so this identifies the set $\{C_2\text{-ring maps } MU_{\mathbb{R}} \to E\}$ with a subset of

$$\{\text{Ring maps } \pi_{*\rho}MU_{\mathbb{R}} \to \pi_{*\rho}E\}$$

This is surprising, as a ring map $MU_{\mathbb{R}} \to E$ determines a ring map

$$\pi_{\bigstar}MU_{\mathbb{R}} \to \pi_{\bigstar}E$$

and $\pi_{\bigstar}MU_{\mathbb{R}}$ has a huge amount of nonzero classes outside of these degrees.

We have also the following expected analogue of Proposition 2.2.8.

Proposition 3.2.8. *Let E be Real-orientable. For any Real orientation*

$$x : MU_{\mathbb{R}} \to E$$

of E, $MU_{\mathbb{R}} \wedge E$ has two canonical Real orientations given by

$$\eta_L : MU_{\mathbb{R}} \simeq S^0 \wedge MU_{\mathbb{R}} \xrightarrow{\eta_{MU_{\mathbb{R}}} \wedge x} MU_{\mathbb{R}} \wedge E$$

and

$$\eta_R : MU_{\mathbb{R}} \simeq MU_{\mathbb{R}} \wedge S^0 \xrightarrow{id \wedge \eta_E} MU_{\mathbb{R}} \wedge E$$

Let $f(x) = x + \sum_{j \geq 1} b_j x^{j+1}$ be the strict isomorphism

$$f : \eta_L^* F \to \eta_R^* F$$

where F is the formal group law over $(MU_{\mathbb{R}})_{*\rho}$ as in Example 3.2.6. Then the map

$$E_{\bigstar}[b_i] \to MU_{\bigstar}E$$

is an isomorphism of $RO(C_2)$-graded $E_{\bigstar}$-algebras, where $|b_i| = i\rho$.

Proof. The non-equivariant proof works with little change. $\square$

We therefore have a Hopf algebroid $((MU_{\mathbb{R}})_{\bigstar}, (MU_{\mathbb{R}} \wedge MU_{\mathbb{R}})_{\bigstar})$, and when restricting to degrees $\bigstar = *\rho$, we have the following

Theorem 3.2.9. *(Landweber-Araki, Hill-Meier) There is an isomorphism of graded Hopf algebroids*

$$((MU_{\mathbb{R}})_{*\rho}, (MU_{\mathbb{R}} \wedge MU_{\mathbb{R}})_{*\rho}) \cong (MU_{2*}, MU_{2*}MU)$$

Proof. The map $MU_{2*} \to (MU_{\mathbb{R}})_{*\rho}$ classifying the formal group law furnished by the Real orientation of $MU_{\mathbb{R}}$ is an isomorphism (see [53] and [3]). The isomorphism of Hopf algebroids is [40, Lemma 3.8]. $\square$

Given this theorem, it should be the case then that for any C_2-spectrum X, we have an associated $\mathbb{G}_m$-equivariant sheaf

$$\mathcal{F}_X \in \mathbf{QCoh}(\mathcal{M}_{FG}(1))$$

via the $(MU_{\mathbb{R}})_{*\rho}, (MU_{\mathbb{R}} \wedge MU_{\mathbb{R}})_{*\rho})$-comodule $(MU_{\mathbb{R}})_{\bigstar}X$. However, we have a splitting of such comodules

$$(MU_{\mathbb{R}})_{\bigstar}X \cong \bigoplus_{i \in \mathbb{Z}} (MU_{\mathbb{R}})_{*\rho+i}(X)$$

and so $(MU_{\mathbb{R}})_{\bigstar}X$ is properly interpreted as giving us a $\mathbb{Z}$-graded $\mathbb{G}_m$-equivariant quasicoherent sheaf $\mathcal{F}_X^i$ on $\mathcal{M}_{FG}(1)$. See [40, Section 3.2] for more details. There is a version of the Landweber exact functor theorem in this context, but it is again slightly more complicated due to the gradings.

Proposition 3.2.10. *(Real Landweber exact functor theorem, Hill-Meier) Let E_{2*} be a graded Landweber exact MU_{2*}-algebra. Then the functor*

$$X \mapsto (MU_{\mathbb{R}})_{\bigstar}(X) \otimes_{MU_{2*}} E_{2*}$$

is a C_2-equivariant homology theory and is therefore represented by a genuine C_2-spectrum.

Proof. This is [40, Theorem 3.6]. We comment on the gradings: as before we have a decomposition of MU_{2*}-modules

$$(MU_{\mathbb{R}})_{\bigstar}(X) \cong \bigoplus_{i\in\mathbb{Z}}(MU_{\mathbb{R}})_{*\rho+i}(X)$$

and we set

$$(MU_{\mathbb{R}})_{\bigstar}(X) \otimes_{MU_{2*}} E_{2*} \coloneqq \bigoplus_{i\in\mathbb{Z}}\left((MU_{\mathbb{R}})_{*\rho+i}(X) \otimes_{MU_{2*}} E_{2*}\right)$$

□

Example 3.2.11. Using the above proposition, the MU_{2*}-algebras BP_{2*} and $E(n)_{2*}$ (here $E(n)$ is at the prime 2) give us Real landweber exact C_2-spectra $BP_{\mathbb{R}}$ and $E_{\mathbb{R}}(n)$. These have the property that $i_e^{C_2}BP_{\mathbb{R}} \simeq BP$ and $i_e^{C_2}E_{\mathbb{R}}(n) \simeq E(n)$, i.e. they are genuine C_2 lifts of BP and $E(n)$. We call $E_{\mathbb{R}}(n)$ the n-th Real Johnson-Wilson theory.

Example 3.2.12. Fix a perfect field k of characteristic 2 and a formal group $\mathbb{G}$ of height n over k. The associated Morava E-theory E_n has the property that $(E_n)_{2*}$ is a Landweber exact MU_{2*}-algebra. Real Landweber exactness thus provides a lift of E_n to a genuine C_2-spectrum. On the other hand, we may regard E_n as a genuine C_2-spectrum as in Example 3.1.12. It is a theorem of Hahn and Shi [34] that these two C_2-spectra are equivalent, and, in particular, we have that E_n is Real-orientable.

3.3 The slice filtration and the HHR slice theorem

In their paper on the Kervaire invariant elements [39], Hill, Hopkins, and Ravenel (HHR) explored the question of whether a similar chromatic-type theory for C_{2^n}-spectra exists, in particular via the C_{2^n}-spectrum $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$. They showed that, surprisingly, $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ behaves very similarly to $MU_{\mathbb{R}}$ and MU. To formulate results in this direction, they introduced the slice filtration of genuine G-spectra. Later, Ullman showed that a slight variant of the HHR slice filtration had slightly better formal properties and agreed with the HHR filtration for the C_{2^n}-spectrum $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ [89]. This filtration is known now as the *regular* slice filtration, and we will use the regular slice filtration in all that follows.

Definition 3.3.1. A *slice sphere* of dimension n is a G-spectrum of the form

$$G_+ \wedge_H S^{m\rho_H}$$

where $m|H| = n$, and ρ_H is the regular representation of H. We say a G-spectrum X is slice $< n$ if the space

$$\mathrm{Map}_{\mathbf{Sp}^G}(S, X)$$

is contractible for all slice spheres S of dimension $\geq n$. We say a G-spectrum X is *slice* $\geq n$ if X is in the localizing subcategory generated by slice spheres of dimension $\geq n$. Here, a *localizing subcategory* $\tau \subset \mathbf{Sp}^G$ is a full subcategory with the following properties:

1. If $X \in \tau$ and $Y \simeq X$, then $Y \in \tau$.

2. If $X \to Y \to Z$ is a cofiber sequence and $X \in \tau$, then $Y \in \tau \iff Z \in \tau$.

3. τ is closed under arbitrary coproducts.

Theorem 3.3.2. *(HHR) For any G-spectrum, there are functorial cofiber sequences*

$$P_{n+1}X \to X \to P^n X$$

such that $P_{n+1}X$ *is slice* $\geq n+1$, $P^n X$ *is slice* $\leq n$, $\mathrm{colim}_n P^n X \simeq *$, *and* $\varprojlim_n P^n X \simeq X$.

We refer to the tower $\{P^n X\}$ as the slice tower of X, and the fiber

$$P^n_n(X) = \mathrm{fib}(P^n X \to P^{n-1} X)$$

as the n-th slice of X. The following characterization of the subcategory of slice $\geq n$ spectra is very useful.

Proposition 3.3.3. *(Hill-Yarnall [42]) A G-spectrum X is slice* $\geq n$ *if and only if the spectrum* $\Phi^H(X)$ *is* $\lfloor \frac{n}{|H|} \rfloor$*-connected for all* $H \subset G$.

Remark 3.3.4. The slice tower of a G-spectrum gives rise to a spectral sequence called the slice spectral sequence of X, which has signature

$$E_2^{s,t} = \pi_{t-s}^G P_t^t X \implies \pi_{t-s}^G X$$

By applying the functor $F(EG_+, -)$ to the slice tower, one has a tower called the homotopy fixed point tower. The spectral sequence associated to this tower has signature

$$E_2^{s,t} = H^s(G; \pi_t^e X) \implies \pi_{t-s}(X^{hG})$$

where $H^s(G; \pi_t^e X)$ is the group cohomology of the G-module

$$\pi_t^e X = \underline{\pi}_t(X)(G/e)$$

This spectral sequence is called the homotopy fixed point spectral sequence (HFPSS) of X, and the natural transformation $(-) \to F(EG_+, -)$ gives a morphism of spectral sequences

$$\mathrm{SliceSS}(X) \to \mathrm{HFPSS}(X)$$

This map, the slice theorem of HHR, and the theorem of Hahn-Shi (see 3.2.12) have contributed to a significant advancement of our understanding of the HFPSS's of Morava E-theories at the prime 2, via pushing forward information in slice spectral sequence along this map. See for example [11] and [41].

We finish the section by stating the slice theorem of HHR, which we state in the BP case rather than MU, as is done in [39]. One may refer to [11, Section 1.3] for a similar presentation of this material. We use the notation used in [39] and define

$$BP^{((C_{2^n}))} := N_{C_2}^{C_{2^n}} BP_{\mathbb{R}}$$

The slice theorem is a description of the slice tower of $BP^{((C_{2^n}))}$, and this begins with a presentation of the underlying homotopy groups that is amenable to the C_{2^n}-action. Let $\mathcal{R}_n$ be the ring $\pi_*^e(BP^{((C_{2^n}))})$, and note that since

$$i_{C_{2^{n-1}}}^{C_{2^n}} BP^{((C_{2^n}))} \simeq BP^{((C_{2^{n-1}}))} \wedge BP^{((C_{2^{n-1}}))}$$

the unit map

$$\eta_L : BP^{((C_{2^{n-1}}))} \simeq S^0 \wedge BP^{((C_{2^{n-1}}))} \to BP^{((C_{2^{n-1}}))} \wedge BP^{((C_{2^{n-1}}))} \simeq i^{C_{2^n}}_{C_{2^{n-1}}} BP^{((C_{2^n}))}$$

gives an inclusion $\mathcal{R}_{n-1} \to \mathcal{R}_n$. By composing, we have an inclusion

$$BP_* \cong \mathcal{R}_1 \hookrightarrow \mathcal{R}_n$$

and we let F be the formal group law defined over $\mathcal{R}_n$ given by pushing forward the universal one over BP_* along this map.

Let γ be a generator of C_{2^n}. We have also the complex orientation

$$BP \xrightarrow{F} i^{C_{2^n}}_{C_{2^{n-1}}} BP^{((C_{2^n}))} \xrightarrow{\gamma} i^{C_{2^n}}_{C_{2^{n-1}}} BP^{((C_{2^n}))}$$

classifying the formal group law we call F^γ, and the two differ by a strict isomorphism ψ as in Lemma 2.2.4, for ψ of the form

$$\psi(x) = x +_{F^\gamma} \sum_{i \geq 1}^{F^\gamma} t_i^{C_{2^n}} x^{2^i}$$

for classes $t_i^{C_{2^n}} \in \pi^e_{2(2^i-1)} BP^{((C_{2^n}))}$. These $t_i^{C_{2^n}}$'s generate $\mathcal{R}_n$ as a C_{2^n}-algebra.

Proposition 3.3.5. *The map of C_{2^n}-algebras*

$$\mathbb{Z}_{(2)}[C_{2^n} \cdot t_1^{C_{2^n}}, C_{2^n} \cdot t_2^{C_{2^n}}, \ldots] \to \mathcal{R}_n$$

is an isomorphism, where $C_{2^n} \cdot x$ represents the set

$$C_{2^n} \cdot x \coloneqq \{x, \gamma(x), \gamma^2(x), \gamma^3(x), \ldots, \gamma^{2^{n-1}-1}(x)\}$$

and $\gamma^{2^{n-1}}(t_i^{C_{2^n}}) = -t_i^{C_{2^n}}$.

Proof. See [39, Section 5.4]. □

It follows from the isomorphism $\mathrm{Res} : (BP_{\mathbb{R}})_{*\rho} \to (BP)_{2*}$ along with Proposition 3.2.8 that the restriction map

$$\pi^{C_2}_{*\rho} BP^{((C_{2^n}))} \to \pi^e_{2*} BP^{((C_{2^n}))} = \mathcal{R}_n$$

is an isomorphism, and we let $\overline{t_i}^{C_{2^n}}$ denote the unique lift of $t_i^{C_{2^n}}$ along this restriction map. $MU^{((C_{2^n}))}$ has the structure of a genuine G-commutative ring (see [39, Section 2.3]), and, in particular, there is a map of C_{2^n}-ring spectra

$$N_{C_2}^{C_{2^n}} i_{C_2}^{C_{2^n}}(MU^{((C_{2^n}))}) \to MU^{((C_{2^n}))}$$

which allows us to define norm classes

$$N_{C_2}^{C_{2^n}}(\overline{t_i}^{C_{2^n}}) \in \pi_{(2^i-1)\rho_{2^n}}^{C_{2^n}}(BP^{((C_{2^n}))})$$

where ρ_{2^n} is the real regular representation of C_{2^n}. Moreover, the map of A_∞-rings

$$S^0[\overline{t_i}^{C_{2^n}}] \to i_{C_2}^{C_{2^n}} BP^{((C_{2^n}))}$$

norms to an A_∞-map

$$A \coloneqq N_{C_2}^{C_{2^n}}(S^0[\overline{t_i}^{C_{2^n}}]) \to BP^{((C_{2^n}))}$$

The A_∞-ring A is a wedge of slice spheres, and we define the monomial ideal $M_{2d} \subset A$ to be the wedge of all such slice spheres of dimension $\geq 2d$, and set

$$K_{2d} \coloneqq BP^{((C_{2^n}))} \wedge_A M_{2d}$$

Theorem 3.3.6. *(HHR slice theorem) One has equivalences*

$$P^{2d+1} BP^{((C_{2^n}))} \simeq P^{2d} BP^{((C_{2^n}))} \simeq BP^{((C_{2^n}))}/K_{2d+2}$$

The odd slices of $BP^{((C_{2^n}))}$ are contractible, and

$$P_{2d}^{2d} BP^{((C_{2^n}))} \simeq H\underline{\mathbb{Z}}_{(2)} \wedge M_{2d}/M_{2d+2}$$

In particular the slice associated graded of $BP^{((C_{2^n}))}$ is

$$H\underline{\mathbb{Z}}_{(2)} \wedge N_{C_2}^{C_{2^n}}(S^0[\overline{t_i}^{C_{2^n}}])$$

Proof. See [39, Section 6]. □

Remark 3.3.7. We explain how one ought to interpret the above theorem. We use the case $n = 2$, i.e $G = C_4$, for concreteness. The slice tower for $BP^{((C_4))}$ forgets to the ordinary Postnikov tower of $i_e^{C_4} BP^{((C_4))}$, which has $P_{2d-1}^{2d-1} \simeq *$ and

$$P_{2d}^{2d} \simeq H\mathbb{Z}_{(2)} \wedge W_{2d}$$

where W_{2d} is a wedge of S^{2d}'s over the set of monomials of degree $2d$ in

$$\pi_*^u(BP^{((C_4))}) = \mathbb{Z}_{(2)}[t_i, \gamma(t_i) : i \geq 1]$$

The slice tower is an equivariant refinement of this wherein the odd slices vanish, $H\mathbb{Z}_{(2)}$ is replaced with $H\underline{\mathbb{Z}}_{(2)}$, the spheres in W_{2d} corresponding to a summand of the above C_4-module with stabilizer C_2 are grouped with their conjugates in a

$$C_{4+} \wedge_{C_2} S^{d\rho_2},$$

the spheres corresponding to a C_4-fixed summand are replaced with $S^{\frac{d}{2}\rho_4}$, and there are no free summands.

3.4 The Segal conjecture for C_p

The Segal conjecture is a fundamental result in equivariant homotopy that gives a calculation of the stable cohomotopy of classifying spaces BG for a finite group. In this section, we use the C_p-Tate square show that the case $G = C_p$ of the Segal conjecture is equivalent to the claim that the Tate diagonal

$$S^0 \to (N_e^{C_p}(S^0))^{tC_p}$$

is a p-complete equivalence. This result can be found in [14]. We motivate the Segal conjecture by recalling the Atiyah-Segal completion theorem.

Theorem 3.4.1. *(Atiyah-Segal) Let KU denote complex topological K-theory, and let G be a compact Lie group. There is an isomorphism of rings*

$$KU^0(BG) \cong R(G)^{\wedge}_I$$

k a field	$\mathbb{F}_1$
Finite-dimensional vector space	Finite Set
$GL_n(k)$	Σ_n
Finite-dimensional representation of G	Finite G-set
$R(G)$	$A(G)$

Table 3.1: Analogy $k : \mathbb{F}_1$

where $R(G)$ is the complex representation ring of G, and I is the augmentation ideal in $R(G)$.

Proof. See [2]. $\square$

The Segal conjecture can be thought of as a generalization of the Atiyah-Segal completion theorem to *algebraic* K-theory. In particular, one can associate an algebraic K-theory spectrum $K(R)$ to a commutative ring R and ask if there is a similar description of the ring $K(R)^0(BG)$. This question has been investigated by a number of authors, and similar completion statements have been proven in some cases; we refer the reader to [68], [87], and [88] to name just a few. The Segal conjecture gives an answer to this question when R is $\mathbb{F}_1$, the field with one element. There is of course no such field, but there is a convincing sense in which the sphere spectrum $\mathbb{S}$ may be thought of as $K(\mathbb{F}_1)$.

When one speaks of the field with one element, $\mathbb{F}_1$, one is referring to a strong analogy between the category of finite sets and the category of finite dimensional vector spaces over a field k, as displayed in Table 3.1. Note here $A(G)$ is the Burnside ring of G.

For any permutative category $\mathcal{C}$, there is an associated algebraic K-theory spectrum $K(\mathcal{C})$. When $\mathcal{C} = \mathbf{FinSets}$, the category of finite sets, one has the following theorem of Barratt-Priddy-Quillen, which makes precise the statement $K(\mathbb{F}_1) \simeq \mathbb{S}$.

Theorem 3.4.2. *Let $\mathcal{C}$ be a skeleton of* **FinSets** *regarded as a permutative category. There* *is an equivalence*

$$K(\mathcal{C}) \simeq \mathbb{S}$$

Proof. See [8]. □

With this in place, the analogy suggests the following theorem of Gunnar Carlsson, known as the Segal conjecture.

Theorem 3.4.3. *Let $\pi^0_{st}(-)$ denote zeroth stable cohomotopy - i.e. the cohomology theory represented by the sphere spectrum $\mathbb{S}$. If G is a finite group, one has an isomorphism*

$$\pi^0_{st}(BG) \cong A(G)\hat{}_I$$

where $A(G)$ is the Burnside ring of G and I is its augmentation ideal.

Proof. See [16]. □

Remark 3.4.4. Carlsson's proof rests on the important base cases $G = C_p$. When $p = 2$, this was proven by Lin in [56], and it was proven by Gunawardena at odd primes [32]. The proofs of these cases involve difficult computations in the Adams spectral sequence, but the description of C_p-equivariant stable homotopy via the C_p Tate square allows for the result in these cases to be recast in several ways, and we focus on the $G = C_p$ case for the rest of the section.

We first identify the zeroth stable cohomotopy group of BG with the zeroth equivariant homotopy group of the Borel sphere.

Lemma 3.4.5. *There is an isomorphism*

$$\pi^0_{st}(BG) \cong \pi^G_0(F(EG_+, S^0))$$

Proof. One has isomorphisms

$$\begin{aligned}
\pi_0^G(F(EG_+, S^0)) &\cong [EG_+, S^0]^G \\
&\cong [EG_+, EG_+]^G \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(\mathbf{Fun}(BG,\mathbf{Sp}))}(S^0, S^0) \\
&\cong [(S^0)_{hG}, S^0] \\
&\cong [BG_+, S^0] \\
&= \pi^0_{st}(BG)
\end{aligned}$$

From the first line to the second, we have used the isotropy separation sequence and the fact that

$$\mathrm{Map}_{\mathbf{Sp}^G}(EG_+, \tilde{E}G) \simeq *$$

The next isomorphism uses the fact that

$$EG_+ \wedge (-) : \mathbf{Fun}(BG, \mathbf{Sp}) \to \mathbf{Sp}^G$$

is a fully faithful left adjoint (see Remark 3.1.11). The next isomorphism uses the fact that

$$(-)_{hG} : \mathbf{Fun}(BG, \mathbf{Sp}) \to \mathbf{Sp}$$

is left adjoint to the trivial action functor $i_* : \mathbf{Sp} \to \mathbf{Fun}(BG, \mathbf{Sp})$. Finally, we make the identifications

$$(S^0)_{hG} \simeq EG_+ \wedge_G S^0 \simeq (EG \times_G *)_+ \simeq BG_+$$

□

Under this isomorphism, using also the isomorphism $\pi_0^G(S^0) \cong A(G)$ of Remark 3.1.10, the Segal conjecture for G becomes the claim that there is an isomorphism

$$\pi_0^G(S^0)\hat{}_I \cong \pi_0^G(F(EG_+, S^0))$$

Homotopy theorists are fond of lifting algebraic completion statements like the one appearing in the Segal conjecture to *topological* completions, and in this case we make the following definition for $G = C_p$.

Definition 3.4.6. Let $(-)^{\hat{}}_I : \mathbf{Sp}^{C_p} \to \mathbf{Sp}^{C_p}$ denote Bousfield localization at the C_p-spectrum $\mathrm{cof}(\epsilon)$, where $\epsilon \in A(C_p)$ is the element $p - [C_p]$, regarded as a map of C_p-spectra

$$\epsilon : S^0 \to S^0$$

Remark 3.4.7. When $G = C_p$, the augmentation ideal I is principal, generated by the above element ϵ. We may therefore think of Bousfield localization at $\mathrm{cof}(\epsilon)$ as topological I-completion. In the following proposition, we give two explicit descriptions of topological I-completion, which serve to justify the terminology.

Proposition 3.4.8. *For $X \in \mathbf{Sp}^{C_p}$, there are equivalences:*

1. $(X)^{\hat{}}_I \simeq \varprojlim_n X/\epsilon^n$ *where X/ϵ^n is the C_p-spectrum given by the cofiber of the map*

$$X \simeq S^0 \wedge X \xrightarrow{\epsilon^n \wedge 1} S^0 \wedge X \simeq X$$

2. $(X)^{\hat{}}_I \simeq F(K(\epsilon), X)$, *where $K(\epsilon)$ fits into a cofiber sequence*

$$K(\epsilon) \to S^0 \to S^0[\epsilon^{-1}]$$

where

$$S^0[\epsilon^{-1}] = \mathrm{colim}(S^0 \xrightarrow{\epsilon} S^0 \xrightarrow{\epsilon} \cdots)$$

Proof. For (1), we first need to establish that $\varprojlim_n X/\epsilon^n$ is $\mathrm{cof}(\epsilon)$-local. Suppose

$$Z \wedge \mathrm{cof}(\epsilon) \simeq *$$

Then

$$\mathrm{Map}_{\mathbf{Sp}^{C_p}}(Z, \varprojlim_n X/\epsilon^n) \simeq \varprojlim_n \mathrm{Map}_{\mathbf{Sp}^{C_p}}(Z, X/\epsilon^n)$$

but $\epsilon^n : Z \to Z$ is an equivalence, so $\mathrm{Map}_{\mathbf{Sp}^{C_p}}(Z, X/\epsilon^n) \simeq *$ for all n. We are done if we can show that

$$X \wedge \mathrm{cof}(\epsilon) \to (\varprojlim_n X/\epsilon^n) \wedge \mathrm{cof}(\epsilon)$$

is an equivalence. This follows from the fact that

$$X/\epsilon^n \wedge \mathrm{cof}(\epsilon) \simeq X \wedge \mathrm{cof}(\epsilon^n) \wedge \mathrm{cof}(\epsilon)$$

and that $\mathrm{cof}(\epsilon^n) \wedge \mathrm{cof}(\epsilon) \simeq \mathrm{cof}(\epsilon)$ for n sufficiently large. Indeed, one has a cofiber sequence

$$\mathrm{cof}(\epsilon) \xrightarrow{\epsilon^n} \mathrm{cof}(\epsilon) \to \mathrm{cof}(\epsilon) \wedge \mathrm{cof}(\epsilon^n)$$

and the second map in the sequence is an equivalence if and only if ϵ^n acts as zero on $\mathrm{cof}(\epsilon)$. We prove this claim in the proof of the lemma below. For (2), taking the colimit in n of cofiber sequences

$$S^0 \xrightarrow{\epsilon^n} S^0 \to S^0/\epsilon^n$$

one has an equivalence $K(\epsilon) \simeq \mathrm{colim}_n \Sigma^{-1} S^0/\epsilon^n$. This gives

$$\begin{aligned} F(K(\epsilon), X) &\simeq F(\mathrm{colim}_n \Sigma^{-1} S^0/\epsilon^n, X) \\ &\simeq \varprojlim_n F(\Sigma^{-1} S^0/\epsilon^n, X) \end{aligned}$$

and the cofiber sequence $S^0 \xrightarrow{\epsilon^n} S^0 \to S^0/\epsilon^n$ gives an equivalence

$$F(\Sigma^{-1} S^0/\epsilon^n, X) \simeq X/\epsilon^n$$

□

Lemma 3.4.9. *There is an isomorphism*

$$\pi_0^{C_p}((S^0)^\wedge_I) \cong \pi_0^{C_p}(S^0)^\wedge_I = A(C_p)^\wedge_I$$

Proof. Since S^0 is connective, we see that

$$\pi_0^{C_p}(S^0/\epsilon^n) \cong \pi_0^{C_p}(S^0)/\epsilon^n = A(C_p)/\epsilon^n$$

By the Milnor sequence, it therefore suffices to show that $\lim^1 \pi_1^{C_p}(S^0/\epsilon^n) = 0$. Note that $\epsilon = p - [C_p]$, and

$$A(C_p) \cong \mathbb{Z}[x]/(x^2 - px)$$

where x corresponds to the C_p-set $[C_p]$. It follows that in $A(C_p)$, $\epsilon^n = p^n - p^{n-1}[C_p]$, so in particular, for an $A(C_p)$-module M,

$$\operatorname{im}(\epsilon^n : M \to M) \subset p^{n-1}M$$

From the long exact sequences in homotopy we have a short exact sequence of pro-$A(C_p)$-modules

$$0 \to \{\operatorname{coker}(\pi_1^{C_p}(S^0) \xrightarrow{\epsilon^n} \pi_1^{C_p}(S^0))\} \to \{\pi_1(S^0/\epsilon^n)\} \to \{\ker(A(C_p) \xrightarrow{\epsilon^n} A(C_p))\} \to 0$$

so it suffices to show the two outer $\lim^1$'s vanish. The right one vanishes because $\ker(\epsilon^n) = \ker(\epsilon)$ for all n - as one checks directly given the above presentation of $A(C_p)$ - so this pro system is constant and the $\lim^1$ vanishes.

For the left one, we use the Tom dieck splitting which implies that

$$(S^0)^{C_p} \simeq S^0 \vee BC_{p_+}$$

so that

$$\pi_1^{C_p}(S^0) = \pi_1((S^0)^{C_p}) = \pi_1(S^0 \vee BC_{p_+}) = \pi_1(S^0) \oplus \pi_1(S^0) \oplus \pi_1(BC_p) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/p$$

This is a stable homotopy group of BC_p but it one coincides with the unstable one because $\mathbb{Z}/p$ is abelian and BC_p is connected. Now we use that

$$\operatorname{im}(\epsilon^n : M \to M) \subset p^{n-1}M$$

If $p = 2$, then we see that ϵ^n acts by zero for $n > 1$, and so the cokernel pro system is constant and the $\lim^1$ vanishes. If $p > 2$, ϵ^n kills the $\mathbb{Z}/p$ component for $n > 1$, and p acts by the identity on the 2-torsion part, so for any x in the 2-torsion part,

$$\epsilon^n(x) = p^n x - p^{n-1}[C_p]x = x - p^{n-1}\operatorname{tr}(\operatorname{res}(x)) = x - \operatorname{tr}(\operatorname{res}(x))$$

using that $\pi_1(S^0) = \mathbb{Z}/2\{\eta\}$ is 2-torsion so the transfer map has to land back in the 2-torsion part. So $\epsilon^n = \epsilon$ as an endomorphism of $\pi_1^{C_p}(S^0)$, so the cokernel pro system is constant and the $\lim^1$ vanishes.

This argument also shows us that ϵ acts nilpotently on $\mathrm{cof}(\epsilon)$: the short exact sequence of $A(C_p)$-modules

$$\mathrm{coker}\left(\pi_1(\mathrm{cof}(\epsilon)) \xrightarrow{\epsilon} \pi_1(\mathrm{cof}(\epsilon))\right) \to [\mathrm{cof}(\epsilon), \mathrm{cof}(\epsilon)]^{C_p} \to \ker\left(\pi_0(\mathrm{cof}(\epsilon)) \xrightarrow{\epsilon} \pi_0(\mathrm{cof}(\epsilon))\right)$$

tells us we need only show ϵ acts nilpotently on $\pi_0(\mathrm{cof}(\epsilon))$ and $\pi_1(\mathrm{cof}(\epsilon))$, which follows easily from the above arguments. □

Using this isomorphism, we may recast the Segal conjecture for C_p in the following way:

Proposition 3.4.10. *The following are equivalent:*

1. *The map $S^0 \to F(EC_{p_+}, S^0)$ is an I-complete equivalence.*
2. *The map $S^0 \to F(EC_{p_+}, S^0)$ is a p-complete equivalence.*
3. *The Tate diagonal $S^0 \to (N_e^{C_p}(S^0))^{tC_p}$ is a p-complete equivalence.*

and each of these implies the Segal conjecture for C_p.

Proof. That the three claims are equivalent follows from applying I-completion to the Tate square for S^0, and using the fact that I-completion and p-completion coincide on C_p-spectra of the form $\tilde{E}C_p \wedge X$, as the ring map

$$S^0 \to \tilde{E}C_p$$

sends $\epsilon = p - [C_p] \mapsto p$.

It suffices to show that the first claim implies the Segal conjecture for C_p, and we begin by showing that $F(EC_{p_+}, S^0)$ is I-complete. By Proposition 3.4.8 (2), it would suffice to show that $S^0[\epsilon^{-1}] \wedge EC_{p_+} \simeq *$. Let $\mathcal{T}$ be the localizing subcategory

$$\mathcal{T} = \{Z \in \mathbf{Sp}^{C_p} : S^0[\epsilon^{-1}] \wedge Z = *\}$$

Then $\mathcal{T}$ contains C_{p_+} because the underlying map of ϵ is 0. It suffices then to note that EC_{p_+} is in the localizing subcategory generated by C_{p_+}, as can be seen from the simplicial filtration of

$$EG_+ \simeq |B(G_+, G_+, *)|$$

Now, if the map $S^0 \to F(EC_{p_+}, S^0)$ is an I-complete equivalence, then

$$(S^0)\hat{}_I \simeq F(EC_{p_+}, S^0)$$

and the Segal conjecture would follow from the previous lemma. □

Remark 3.4.11. Using Proposition 3.4.10 (3), Nikolaus and Scholze considered the Tate diagonal

$$X \to (N_e^{C_p}(X))^{tC_p}$$

and showed that it is a p-complete equivalence for all X bounded below (see [74, III.1.7]). Their argument was by induction up the Postnikov tower of X, whereby they reduced the general case to the case $X = H\mathbb{F}_p$. Using the Tate square for $N_e^{C_p} H\mathbb{F}_p$, this is equivalent to the map of C_p-spectra

$$N_e^{C_p}(H\mathbb{F}_p) \to F(EC_{p_+}, N_e^{C_p}(H\mathbb{F}_p))$$

being an equivalence. We return to this characterization in Chapter 5 at $p = 2$ and use it to give a proof of the C_2-Segal conjecture via Norms of Real bordism theory.

Chapter 4

SMASHING LOCALIZATIONS IN EQUIVARIANT STABLE HOMOTOPY

In this chapter, we investigate analogs of the Ravenel conjectures (see Section 2.2.3) in $\mathbf{Sp}^{C_{2^n}}$, the category of genuine C_{2^n}-spectra. We focus, in particular on the smash product and chromatic convergence theorems. We show that the analogues of these theorems for the $E_{\mathbb{R}}(n)$'s do *not* hold in genuine C_2-spectra, but they do hold in cofree C_2-spectra. That we need to pass to Borel C_2-spectra is perhaps unsurprising because $MU_{\mathbb{R}}$ itself is cofree: it is a theorem of Hu and Kriz that the map

$$MU_{\mathbb{R}} \to F(EC_{2+}, MU_{\mathbb{R}})$$

is an equivalence in $\mathbf{Sp}^{C_2}$, and similarly for $E_{\mathbb{R}}(n)$. In the case of the nilpotence and thick subcategory theorems, the analogs for Real bordism theory are easily seen to fail in genuine C_2-spectra. Passing to Borel C_2-spectra, the nilpotence theorem still fails for $MU_{\mathbb{R}}$, but the analog of the thick subcategory theorem is more delicate, and we remark on the difficulties in Remark 4.3.3.

In their solution to the Kervaire Invariant One problem [39], Hill, Hopkins, and Ravenel construct genuine C_{2^n}-spectra $MU^{((C_{2^n}))} := N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ that bring Real bordism theory into the C_{2^n}-equivariant context. These play an essential role in their proof: their detecting spectrum is a localization of $MU^{((C_8))}$. Recently, Beaudry, Hill, Shi, and Zeng have constructed versions of Johnson-Wilson theories in this context [11], which they call $D^{-1}BP^{((G))}\langle m\rangle$. We give a description of the Bousfield classes of these spectra and deduce analogous results.

Our analysis begins with the observation that the $E_{\mathbb{R}}(n)$'s and $E_G(m)$'s are Bousfield equivalent to certain induced G-spectra. We therefore study in general how Bousfield classes in the equivariant context behave under various change of group functors, most of which send smashing Bousfield classes to smashing Bousfield classes. The exceptional case is that of the induction functor

$$G_+ \wedge_H (-) : \mathbf{Sp}^H \to \mathbf{Sp}^G$$

for a subgroup $H \subset G$. We give a necessary and sufficient condition for a smashing Bousfield class to be preserved by induction, and we find that the above formula for $L_{E_{\mathbb{R}}(n)}$ is generic in this context.

In Section 4.1, we review Bousfield localization of G-spectra and the relationship between smashing localizations and tensor idempotents. In 4.2, we study the interaction between Bousfield localization functors and change of group functors in general, specializing then to smashing localizations.

From here, we move to applications of Section 4.2, beginning in 4.3 with the proofs of the analogues of the smash product and chromatic convergence theorems in the Borel equivariant context - i.e. in cofree G-spectra - and a remark on analogs of the nilpotence and thick subcategory theorems. Nonequivariantly, the functors $L_{E(n)}$ have the additional remarkable property that the subcategories of finite p-local spectra

$$\mathcal{C}_{\geq n} = \{X \in \mathbf{Sp}^{\omega}_{(p)} \; : \; L_{E(n-1)}(X) = 0\}$$

along with the trivial subcategory $\{0\}$ form a complete list of the thick tensor ideals in the category of finite p-local spectra (see Section 2.2.3). A description of the thick tensor ideals in finite G-spectra has been given for all G abelian by [9], but not all of the thick tensor ideals in finite C_2-spectra correspond to $L_{E_{\mathbb{R}}(n)}$ in an analogous way. For $G = C_{p^n}$, we construct a family of new G-spectra $E(\mathcal{J})$ - indexed by the thick tensor ideals $\mathcal{J}$ in $(\mathbf{Sp}^G)^{\omega}_{(p)}$ - such that $L_{E(\mathcal{J})}$ is smashing, $\mathcal{J}$ is the collection of finite acyclics of $E(\mathcal{J})$, and the geometric fixed points of $E(\mathcal{J})$ at any subgroup is a nonequivariant $E(n)$.

In Section 4.4, we use formulae like the above for $L_{E_{\mathbb{R}}(n)}$ to observe that induced localizations upgrade the norms available in an N_∞-algebra, and we determine exactly which new norms appear. This generalizes a result of Blumberg and Hill that if $E \in \mathbf{Sp}^G$ is a cofree

E_∞-ring, it is automatically genuine G-E_∞.

Finally, in Section 4.5, we return to the Borel perspective on the main theorems mentioned above. It is a result of [5] (upgraded to the level of symmetric monoidal ∞-categories by [66]) that $\mathbf{Sp}$, the category of nonequivariant spectra, is equivalent to the category of modules in $\mathbf{Sp}^{C_2}$ over the E_∞-ring $A = F(C_{2+}, S^0)$, so that the coinduction functor becomes restriction of scalars, and the restriction functor becomes extension of scalars. Moreover, extension of scalars induces an equivalence between the category of Borel C_2-spectra and $(\mathrm{Mod}_{\mathbf{Sp}^{C_2}}(A))^{hC_2}$.

We show that, by analogy, if $\eta : \mathbb{1} \to A$ is a quasi-Galois extension in a symmetric monoidal stable ∞-category, it is often possible to use a norm construction to take a smashing A-module M and produce a smashing object in the category of A-locals. This is equivalent to producing a smashing-then-complete type localization formula for $\eta^* M$. We give a necessary and sufficient condition for this localization to be smashing in the category of A-modules.

4.1 Equivariant Bousfield classes

In this section, we review what we need from equivariant Bousfield localization following [37] and smashing localizations following [6].

4.1.1 Equivariant categories of acyclics

We begin with a review of the characterization of acyclics in an equivariant context given in [37].

Definition 4.1.1. If E is a G-spectrum, we let $\mathcal{Z}_E$ denote the category of *E-acyclics*: the full subcategory of $\mathbf{Sp}^G$ consisting of all Z such that $E \wedge Z$ is equivariantly contractible. We let $\mathcal{L}_E$ denote the category of *E-locals*: the full subcategory of $\mathbf{Sp}^G$ consisting of all X such that $\mathbf{Sp}^G(Z, X) \simeq *$ for all $Z \in \mathcal{Z}_E$. We say $E, F \in \mathbf{Sp}^G$ are *Bousfield equivalent* (denoted $\langle E \rangle = \langle F \rangle$) if $\mathcal{Z}_E = \mathcal{Z}_F$.

Since the geometric fixed point functors Φ^H are symmetric monoidal and jointly conservative, this gives us a concrete way to describe $\mathcal{Z}_E$:

Proposition 4.1.2. *[37, Proposition 3.2] If $Z \in \mathbf{Sp}^G$, then $Z \in \mathcal{Z}_E$ if and only if $\Phi^H(Z) \in \mathcal{Z}_{\Phi^H(E)}$ for all subgroups $H \subset G$:*

$$\mathcal{Z}_E = \bigcap_{H \subset G} (\Phi^H)^{-1}(\mathcal{Z}_{\Phi^H(E)})$$

Corollary 4.1.3. *[37] Suppose $E \in \mathbf{Sp}^G$ has the property that $\Phi^H(E) \simeq *$ for all $H \subset G$ nontrivial, then $\mathcal{Z}_E = (\Phi^{\{e\}})^{-1}(\mathcal{Z}_{\Phi^{\{e\}}(E)})$. That is, $Z \in \mathbf{Sp}^G$ is E-acyclic if and only if its underlying spectrum is $\Phi^{\{e\}}(E)$-acyclic.*

From this, we deduce a useful characterization of the Bousfield classes of the Real Johnson-Wilson theories introduced by Hu-Kriz [48] and studied extensively by Kitchloo-Wilson [50].

Example 4.1.4. Let $E_{\mathbb{R}}(n)$ denote the n-th Real Johnson-Wilson theory, E_n a Morava E-theory associated to a perfect field k of characteristic 2 and $\mathbb{G}$ a height n formal group over k, and $E(n)$ the usual nonequivariant Johnson-Wilson theory as in 3.2.12 and 2.2.17. Then

$$\langle E_{\mathbb{R}}(n) \rangle = \langle E_n \rangle = \langle C_{2+} \wedge E(n) \rangle$$

Proof. These three C_2-spectra all have contractible geometric fixed points, and the Bousfield classes of their underlying spectra agree. □

4.1.2 Smashing spectra and idempotent triangles

We review the theory of smashing localizations - for more details see [6], [58], [78], and [79]. We first recall the following basic fact about Bousfield localization that we will use repeatedly. For a reference on the existence of Bousfield localizations of G-spectra, see XXII.6 in [61].

Lemma 4.1.5. *If $E \in \mathbf{Sp}^G$ is a ring spectrum, then any module M over E (e.g. E itself) is E-local.*

Proof. Let $Z \in \mathcal{Z}_E$, then any map $f : Z \to M$ factors as follows

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d] & M \\ E \wedge Z \arrow[r, "1_E \wedge f"] & E \wedge M \arrow[u, "\mu_M"'] \end{tikzcd}$$

but then $E \wedge Z \simeq *$, hence f is null. □

Definition 4.1.6. For $E \in \mathbf{Sp}^G$, let L_E denote the corresponding Bousfield localization functor. We say that L_E is a *smashing localization* or that E is a *smashing* G-spectrum if the natural map

$$L_E(S^0) \wedge X \to L_E(S^0) \wedge L_E(X) \to L_E(X)$$

is an equivalence for all $X \in \mathbf{Sp}^G$.

Recall that Bousfield localization at E determines for each $X \in \mathbf{Sp}^G$ a cofiber sequence

$$Z_E(X) \xrightarrow{\psi_X} X \xrightarrow{\phi_X} L_E(X)$$

with $Z_E(X) \in \mathcal{Z}_E$ and $L_E(X) \in \mathcal{L}_E$, which is unique up to homotopy with respect to these properties.

Proposition 4.1.7. *The following characterizations of smashing localizations are equivalent:*

1. L_E *is smashing.*
2. $\mathcal{L}_E$ *is closed under homotopy colimits.*
3. $\mathcal{L}_E$ *is closed under arbitrary coproducts.*
4. $\mathcal{L}_E$ *is a smash ideal. That is* $X \in \mathcal{L}_E$, $Y \in \mathbf{Sp}^G \implies X \wedge Y \in \mathcal{L}_E$.
5. *If* $R \in \mathcal{L}_E$ *is a ring spectrum, every* R*-module is in* $\mathcal{L}_E$.
6. $\langle E \rangle = \langle L_E(S^0) \rangle$

Proof. For 1 $\iff$ 2 $\iff$ 3 see [58]. We show 1 $\implies$ 4 $\implies$ 5 $\implies$ 6 $\implies$ 1: If L_E is smashing, then if $X \in \mathcal{L}_E$ and $Y \in \mathbf{Sp}^G$,

$$X \wedge Y \simeq L_E(X) \wedge Y \simeq L_E(S^0) \wedge X \wedge Y \simeq L_E(X \wedge Y) \in \mathcal{L}_E$$

If $\mathcal{L}_E$ is a smash ideal, $R \in \mathcal{L}_E$ is a ring spectrum, and M is an R-module, then M is a retract of $R \wedge M$, which must be local, and $\mathcal{L}_E$ is closed under retracts. Note that L_E is lax monoidal (on the level of the homotopy category), hence $L_E(S^0)$ is a ring spectrum in $\mathcal{L}_E$. $\mathcal{Z}_{L_E(S^0)} \subset \mathcal{Z}_E$ is clear, and assuming (5), $Z \in \mathcal{Z}_E$ implies that $Z \wedge L_E(S^0) \in \mathcal{Z}_E$, and as a module over $L_E(S^0)$, $Z \wedge L_E(S^0) \in \mathcal{L}_E$, hence $Z \wedge L_E(S^0) \simeq *$, i.e. $Z \in \mathcal{Z}_{L_E(S^0)}$. Now, since for any $X \in \mathbf{Sp}^G$, $X \to L_E(S^0) \wedge X$ becomes an equivalence after smashing with E, to show L_E is smashing, it suffices to show $L_E(S^0) \wedge X \in \mathcal{L}_E$. But since $L_E(S^0)$ is a ring spectrum, $L_E(S^0) \wedge X \in \mathcal{L}_{L_E(S^0)}$ by 4.1.5, but $\mathcal{L}_{L_E(S^0)} = \mathcal{L}_E$, assuming (6). $\square$

We will prefer characterization (6), as it is the only one that is phrased as a condition on the category of E-acyclics, rather than E-locals. Smashing localizations were studied in a more general setting by Balmer and Favi in [6], and we recall here some of their definitions and results.

Definition 4.1.8. [6, Definition 3.2] Let $(\mathcal{T}, \otimes, \mathbb{1})$ be a tensor-triangulated (tt-) category (e.g. $Ho(\mathbf{Sp}^G)$). We say that a distinguished triangle in $\mathcal{T}$ of the form

$$e \xrightarrow{\psi} \mathbb{1} \xrightarrow{\phi} f \to \Sigma e$$

is an *idempotent triangle* if it satisfies any of the following equivalent conditions:

1. $e \otimes f = 0$
2. $(1_e \otimes \psi) : e \otimes e \to e$ is an isomorphism. (Left Idempotent)
3. $(1_f \otimes \phi) : f \to f \otimes f$ is an isomorphism. (Right Idempotent)

The relationship between idempotent triangles and smashing localizations is as follows.

Definition 4.1.9. [6] Let $\mathcal{T}$ be a tt-category and $\mathcal{J} \subset \mathcal{T}$ a thick tensor ideal. We define

$$\mathcal{J}^{\perp} = \{t \in \mathcal{T} : \mathrm{Hom}_{\mathcal{T}}(z,t) = 0 \text{ for all } z \in \mathcal{J}\}$$

We say that $\mathcal{J}$ is a *Bousfield ideal* if for every $t \in \mathcal{T}$, there exists a distinguished triangle

$$e_t \to t \to f_t \to \Sigma e_t$$

such that $e_t \in \mathcal{J}$ and $f_t \in \mathcal{J}^{\perp}$. We say that $\mathcal{J}$ is a smashing ideal if $\mathcal{J}^{\perp}$ is a tensor ideal.

Theorem 4.1.10. *If $(\mathcal{T}, \otimes, \mathbb{1})$ is a rigidly-compactly generated tt-category, there is a 1-1 correspondence between isomorphism classes of idempotent triangles and smashing ideals in $\mathcal{T}$, wherein $\mathcal{J}$ as above corresponds to the triangle*

$$e_{\mathbb{1}} \to \mathbb{1} \to f_{\mathbb{1}} \to \Sigma e_{\mathbb{1}}$$

and an idempotent triangle

$$e \to \mathbb{1} \to f \to \Sigma e$$

corresponds to the smashing ideal $\ker(- \otimes f)$.

Proof. See [6, Theorem 3.5]. □

Corollary 4.1.11. *If $(\mathcal{T}, \otimes, \mathbb{1}) = (Ho(\mathbf{Sp}^G), \wedge, S^0)$, there is a 1-1 correspondence between isomorphism classes of idempotent triangles in $\mathcal{T}$ and smashing Bousfield classes $\langle E \rangle$, and hence also between smashing ideals in $Ho(\mathbf{Sp}^G)$ and smashing Bousfield classes.*

Proof. Each smashing $\langle E \rangle$ determines the idempotent triangle

$$Z_E(S^0) \to S^0 \to L_E(S^0) \to \Sigma Z_E(S^0)$$

as $L_E(S^0) \wedge Z_E(S^0) \simeq L_E(Z_E(S^0)) \simeq *$. Conversely, if $e \to S^0 \to f \to \Sigma e$ is an idempotent triangle, then it follows that it is isomorphic to

$$Z_f(S^0) \to S^0 \to L_f(S^0) \to \Sigma Z_f(S^0)$$

and therefore corresponds to the smashing Bousfield class $\langle f \rangle$. Indeed, f is a ring spectrum via the isomorphism $f \otimes f \cong f$, so f is f-local, and the map $S^0 \to f$ is therefore isomorphic as a right idempotent to the map $S^0 \to L_f(S^0)$. These give mutually inverse maps of posets because if $\langle E \rangle$ is smashing, then $\langle E \rangle = \langle L_E(S^0) \rangle$ by 4.1.7, and conversely we have just shown that $f \cong L_f(S^0)$. □

Example 4.1.12. Let $\mathcal{F}$ be a family of subgroups of G and $E\mathcal{F}$ the corresponding universal space. It is easy to see that $E\mathcal{F} \times E\mathcal{F}$ is also a universal space for $\mathcal{F}$, hence the collapse map

$$E\mathcal{F}_+ \wedge E\mathcal{F}_+ \to E\mathcal{F}_+ \wedge S^0 \simeq E\mathcal{F}_+$$

is an equivalence, so that

$$E\mathcal{F}_+ \to S^0 \to \tilde{E}\mathcal{F} \to \Sigma E\mathcal{F}_+$$

is an idempotent triangle corresponding to the smashing localization $L_{\tilde{E}\mathcal{F}}(-)$.

Corollary 4.1.13. *If $E_1, \ldots, E_n \in \mathbf{Sp}^G$ are all smashing, then so are $E_1 \wedge \cdots \wedge E_n$ and $E_1 \vee \cdots \vee E_n$. Moreover, $Z_{E_1 \vee \cdots \vee E_n}(S^0) \simeq Z_{E_1}(S^0) \wedge \cdots \wedge Z_{E_n}(S^0)$ and $L_{E_1 \wedge \cdots \wedge E_n}(S^0) \simeq L_{E_1}(S^0) \wedge \cdots \wedge L_{E_n}(S^0)$.*

Proof. It is shown in [6] that the tensor product gives the product in the category of left idempotents and the coproduct in the category of right idempotents. It follows from 4.1.10 then that the poset of smashing ideals in $\mathbf{Sp}^G$ has meets and joins, and if E, F are smashing G-spectra, these correspond to $E \vee F$ and $E \wedge F$ respectively. □

4.2 Bousfield localizations and change of group

In this section, we start with a G-spectrum E and explore the Bousfield localization functors associated to the spectrum $F(E)$ along various change of group functors F. We explore whether $F(E)$ is smashing, assuming that E is smashing.

4.2.1 *The general case*

We first establish some elementary facts about the behavior of localization functors along change of group functors F in general. We first recall some definitions from Sectionsubsec3.1.2

Definition 4.2.1. Let $i_* : \mathbf{Sp} \to \mathbf{Sp}^G$ denote the functor that sends a spectrum to the corresponding G-spectrum with trivial action, and $(-)^G : \mathbf{Sp}^G \to \mathbf{Sp}$ its right adjoint, the genuine fixed points. For a subgroup $H \subset G$, let $i_H^G : \mathbf{Sp}^G \to \mathbf{Sp}^H$ and $G_+ \wedge_H (-) : \mathbf{Sp}^H \to \mathbf{Sp}^G$ denote the restriction and induction functors respectively.

Proposition 4.2.2. *For any $E, X \in \mathbf{Sp}^G$, we have*

$$L_{i_H^G E}(i_H^G X) \simeq i_H^G L_E(X)$$

Proof. i_H^G is symmetric monoidal, hence the map $i_H^G X \to i_H^G L_E X$ becomes an equivalence after smashing with $i_H^G E$. $i_H^G L_E(X)$ is $i_H^G E$-local because if $Z \in \mathcal{Z}_{i_H^G E}$, then

$$[Z, i_H^G L_E(X)]^H \cong [G_+ \wedge_H Z, L_E(X)]^G$$

and $G_+ \wedge_H Z \in \mathcal{Z}_E$, as

$$(G_+ \wedge_H Z) \wedge E \simeq G_+ \wedge_H (Z \wedge i_H^G E) \simeq *$$

□

From 4.1.2, it is not difficult in general to characterize the $F(E)$-acyclics in terms of the E-acyclics, where F is one of our change of group functors above. Characterizing the $F(E)$-locals in terms of the E-locals is much more difficult. For restriction and induction, however, we can give a simple necessary and sufficient condition.

Proposition 4.2.3. *For any $E \in \mathbf{Sp}^G$, $Y \in \mathbf{Sp}^H$ is $i_H^G E$-local if and only if $G_+ \wedge_H Y$ is E-local.*

Proof. If Y is $i_H^G E$-local, then if $Z \in \mathcal{Z}_E$, we have

$$[Z, G_+ \wedge_H Y]^G \cong [i_H^G Z, Y]^H = 0$$

via the Wirthmüller isomorphism. Conversely, if $Z \in \mathcal{Z}_{i_H^G E}$, $G_+ \wedge_H \mathcal{Z}_{i_H^G E} \subset \mathcal{Z}_E$ implies that

$$[Z, i_H^G(G_+ \wedge_H Y)]^H \cong [G_+ \wedge_H Z, G_+ \wedge_H Y]^G = 0$$

and since Y is a summand of $i_H^G(G_+ \wedge_H Y)$, $[Z, Y]^H = 0$. □

Definition 4.2.4. Let $H \subset G$, then we let $\mathcal{F}_H$ be the family of subgroups of G that are subconjugate to H - that is, $\mathcal{F}_H$ is the smallest family of subgroups of G containing H. We say a G-spectrum X is H*-cofree* if the canonical map

$$X \to F(E\mathcal{F}_{H+}, X)$$

is an equivalence, where $F(-,-)$ denotes the internal mapping spectrum in $\mathbf{Sp}^G$, and $E\mathcal{F}_H$ is the universal G-space for the family $\mathcal{F}_H$. We simply say cofree, or Borel complete, when $H = \{e\}$. See also Remark 3.1.11.

Lemma 4.2.5. *If $X \in \mathbf{Sp}^G$, then $F(E\mathcal{F}_{H+}, X) \simeq L_{G/H_+}(X)$.*

Proof. Since $i_H^G(E\mathcal{F}_{H+}) \simeq S^0$, it follows that

$$X \to F(E\mathcal{F}_{H+}, X)$$

becomes an equivalence after smashing with G/H_+. $F(E\mathcal{F}_{H+}, X)$ is G/H_+-local because if $Z \in \mathcal{Z}_{G/H_+}$ so that $i_H^G Z \simeq *$, then

$$[Z, F(E\mathcal{F}_{H+}, X)]^G \cong [Z \wedge E\mathcal{F}_{H+}, X]^G$$

and $Z \wedge E\mathcal{F}_{H+} \simeq *$. For this, let

$$\mathcal{T} = \{Y \in \mathbf{Sp}^G : Z \wedge Y \simeq *\}$$

then $\mathcal{T}$ is a localizing subcategory of $\mathbf{Sp}^G$, and $E\mathcal{F}_{H+}$ is in the localizing subcategory generated by $\{G/K_+ : K \in \mathcal{F}_H\}$, so it suffices to observe that $G/K_+ \in \mathcal{T}$ for all $K \in \mathcal{F}_H$. □

Corollary 4.2.6. *A map $f : X \to Y$ in $\mathbf{Sp}^G$ between H-cofree G-spectra is an equivalence if and only if $i_H^G(f)$ is an equivalence.*

Proof. In general, a map between E-locals is an equivalence if and only if it becomes an equivalence after smashing with E. Letting $E = G/H_+$ gives the result. □

Proposition 4.2.7. *For any $E \in \mathbf{Sp}^H$, $X \in \mathbf{Sp}^G$ is $G_+ \wedge_H E$-local if and only if X is H-cofree and $i_H^G X$ is $i_H^G(G_+ \wedge_H E)$-local.*

Proof. Suppose X is $G_+ \wedge_H E$-local. Clearly, $\mathcal{Z}_{G/H_+} \subset \mathcal{Z}_{G_+ \wedge_H E}$ and hence $\mathcal{L}_{G_+ \wedge_H E} \subset \mathcal{L}_{G/H_+}$ - that is, $G_+ \wedge_H E$-locals are H-cofree. $i_H^G X$ is $i_H^G(G_+ \wedge_H E)$-local by 4.2.2.

Conversely, if X is H-cofree, then it suffices to show the map $\phi_X : X \to L_{G_+ \wedge_H E}(X)$ is an equivalence, and by 4.2.6, it suffices to show that $i_H^G(\phi_X)$ is an equivalence, which follows again by assumption from 4.2.2. □

Remark 4.2.8. Since E is a retract of $i_H^G(G_+ \wedge_H E)$, we have $\mathcal{Z}_{i_H^G(G_+ \wedge_H E)} \subset \mathcal{Z}_E$ and hence $\mathcal{L}_E \subset \mathcal{L}_{i_H^G(G_+ \wedge_H E)}$. For X to be $G_+ \wedge_H E$-local, it is therefore *sufficient* for X to be H-cofree and $i_H^G X$ to be E-local.

The following are easy consequences of the double coset formula for $i_H^G(G_+ \wedge_H E)$.

Corollary 4.2.9. *If $H \subset G$ is normal, $X \in \mathbf{Sp}^G$ is $G_+ \wedge_H E$-local if and only if X is H-cofree and $i_H^G X$ is $\bigvee_{[g] \in G/H} {}^g E$-local, where ${}^g E$ are the Weyl conjugates of E.*

Example 4.2.10. We give an example showing that E and ${}^g E$ do not necessarily determine the same Bousfield class. Let $G = \Sigma_4$, $H = V_4 \triangleleft G$, and $g = (123)$. Then

$$ {}^g \tilde{E}\mathcal{F}_{\langle (12)(34) \rangle} \simeq \tilde{E}\mathcal{F}_{\langle (14)(23) \rangle} $$

It is straightforward to check on geometric fixed points that $E\mathcal{F}_{\langle (12)(34) \rangle_+} \notin \mathcal{Z}_{\tilde{E}\mathcal{F}_{\langle (14)(23) \rangle}}$ but of course $E\mathcal{F}_{\langle (12)(34) \rangle_+} \in \mathcal{Z}_{\tilde{E}\mathcal{F}_{\langle (12)(34) \rangle}}$.

Corollary 4.2.11. *If G is abelian, $X \in \mathbf{Sp}^G$ is $G_+ \wedge_H E$-local if and only if X is H-cofree and $i_H^G X$ is E-local.*

4.2.2 The smashing case

We now discuss how smashing localizations behave under change of group functors. We first recall the following variant of the norm functor $N_H^G : \mathbf{Sp}^H \to \mathbf{Sp}^G$ of [39]. Let $N^{G/H} : \mathbf{Sp}^G \to \mathbf{Sp}^G$ denote the composition $N_H^G \circ i_H^G$, and for

$$T = G/H_1 \sqcup \cdots \sqcup G/H_n$$

a finite G-set, we let $N^T : \mathbf{Sp}^G \to \mathbf{Sp}^G$ denote the functor $N^{G/H_1} \wedge \cdots \wedge N^{G/H_n}$. We will also need the following description of how geometric fixed points interact with the norm.

Proposition 4.2.12. *For any $K, H \subset G$, and for any $E \in \boldsymbol{Sp}^H$, the diagonal gives an equivalence of spectra*

$$\Phi^K N_H^G E \xrightarrow{\simeq} \bigwedge_{[g] \in K \backslash G/H} \Phi^{K^g \cap H} E$$

Proof. See [39, Proposition B.209]. □

Proposition 4.2.13. *Let $H \subset G$ be a subgroup. Smashing Bousfield classes are preserved by the following change of group functors:*

1. *If $E \in \mathbf{Sp}$ is smashing, then $i_* E \in \mathbf{Sp}^G$ is smashing.*

2. *If $E \in \mathbf{Sp}^G$ is smashing, then $i_H^G E \in \mathbf{Sp}^H$ is smashing.*

3. *If $E \in \mathbf{Sp}^G$ is smashing, then $\Phi^H(E) \in \mathbf{Sp}$ is smashing.*

4. *Let $f : G \to G'$ be a group homomorphism and $f^* : \mathbf{Sp}^{G'} \to \mathbf{Sp}^G$ the induced functor. If $E \in \mathbf{Sp}^{G'}$ is smashing, then $f^* E \in \mathbf{Sp}^G$ is smashing.*

5. *If $E \in \mathbf{Sp}^H$ is smashing, then $N_H^G E$ is smashing.*

6. *If $E \in \mathbf{Sp}^G$ is smashing, and T is a finite G-set, then $N^T E$ is smashing.*

7. *If $E \in \mathbf{Sp}^G$ is smashing, and for all $H \subset G$, $\mathcal{Z}_{E^G} \subset \mathcal{Z}_{E^H}$ (e.g. if E is a ring spectrum), then E^G is smashing.*

Moreover, for each functor F in items (1)-(6), we have $L_{F(E)}(F(X)) \simeq F(L_E(X))$. In item (7), we have

$$Z_{E^G}(X) = \bigwedge_{H \subset G} \Phi^H(Z_E(X))$$

so that

$$L_{E^G}(X) = \mathrm{cofib}\Bigg(\bigwedge_{H \subset G} \Phi^H(Z_E(X)) \to S^0\Bigg)$$

Proof. In all cases, we have a smashing spectrum E and therefore $\langle E \rangle = \langle L_E(S^0) \rangle$. If F is one of the functors listed in items (1)-(6), it is symmetric monoidal, and hence $F(L_E(S^0))$ is a right idempotent, so it suffices to show $\langle F(L_E(S^0)) \rangle = \langle F(E) \rangle$. We therefore show, more generally, that if $\langle X \rangle = \langle Y \rangle$, then $\langle F(X) \rangle = \langle F(Y) \rangle$ if F is one of the functors listed in items (1)-(6). For (1), the relation $\Phi^H \circ i_* \simeq \mathrm{id}_{\mathbf{Sp}}$ for all $H \subset G$ gives

$$\langle \Phi^H(i_* X) \rangle = \langle X \rangle = \langle Y \rangle = \langle \Phi^H(i_* Y) \rangle \implies \langle i_* X \rangle = \langle i_* Y \rangle$$

For (2), note that

$$Z \in \mathcal{Z}_{i_H^G X} \iff (G_+ \wedge_H Z) \wedge X \simeq * \iff (G_+ \wedge_H Z) \wedge Y \simeq *$$

$i_H^G Y$ is shown to have the same acyclics by an identical argument. For (3), we have

$$\begin{aligned} Z \wedge \Phi^G(X) \simeq * &\iff \tilde{E}\mathcal{P} \wedge i_* Z \wedge X \simeq * \\ &\iff \tilde{E}\mathcal{P} \wedge i_* Z \wedge Y \simeq * \\ &\iff Z \wedge \Phi^G(Y) \simeq * \end{aligned}$$

For (4), if H is any subgroup of G', the relation $\Phi^H \circ f^* = \Phi^{f(H)}$ gives

$$\langle \Phi^H(f^* X) \rangle = \langle \Phi^{f(H)} X \rangle = \langle \Phi^{f(H)}(Y) \rangle = \langle \Phi^H(f^* Y) \rangle$$

by applying case (3). For (5), 4.2.12 gives

$$\begin{aligned}\langle \Phi^K(N_H^G(X))\rangle &= \langle \bigwedge_{[g]\in K\backslash G/H} \Phi^{K^g\cap H}(X)\rangle \\ &= \bigwedge_{[g]\in K\backslash G/H} \langle \Phi^{K^g\cap H}(X)\rangle \\ &= \bigwedge_{[g]\in K\backslash G/H} \langle \Phi^{K^g\cap H}(Y)\rangle \\ &= \langle \Phi^K(N_H^G(Y))\rangle\end{aligned}$$

for any subgroup $K \subset G$, again by applying case (3). For (6), if $T = G/H$, the result follows by combining cases (2) and (5), and the general case follows from 4.1.13. The final remark follows in these cases from the localizations being smashing, as then the condition may be checked on the sphere spectrum.

For the genuine fixed points functor (7), we have $\mathcal{Z}_{E^G} \subset Z_{E^H}$ for all $H \subset G$, by assumption, hence

$$\begin{aligned} Z \in \mathcal{Z}_{E^G} &\iff Z \in \mathcal{Z}_{E^H} \ \forall\ H \subset G \\ &\iff (i_* Z \wedge E)^H \simeq * \ \forall\ H \subset G \\ &\iff i_* Z \in \mathcal{Z}_E \end{aligned}$$

and this holds if and only if $Z \wedge \Phi^H(E) \simeq *$ for all $H \subset G$. For the second equivalence, we use the fact that the natural map

$$Z \wedge E^H \to i_*(Z \wedge E)^H$$

is an equivalence (see [37, 3.20]). We find:

$$\langle E^G\rangle = \langle \bigvee_{H\subset G} \Phi^H(E)\rangle$$

and the claim follows as in 4.1.13. The assumptions hold for E a ring spectrum because one has restriction ring maps $E^G \to E^H$ given by applying $(-)^G$ to the map of rings

$$E \to F(G/H_+, S^0) \wedge E$$

□

Remark 4.2.14. We needed to assume E is smashing in 4.2.13 to establish that $\Phi^G(L_E(S^0))$ is $\Phi^G(E)$-local, whereas with i_H^G, we could exploit the existence of a left adjoint to get around this assumption. In fact, it is not necessarily true that $\Phi^G(L_E(S^0))$ is $\Phi^G(E)$-local without this assumption. For example, if $G = C_2$, $E = C_{2+}$, $X = S^0$, then the left hand side is a point, and the right hand is $(S^0)^{tC_2}$. This example also shows us that the converse to case (3) of 4.2.13 is false, i.e. we cannot detect whether E is smashing just by knowing that $\Phi^H E$ is smashing for all $H \subset G$.

Corollary 4.2.15. *We have the following characterizations of local objects for smashing localizations:*

1. *If $E \in \mathbf{Sp}^G$ is smashing, $X \in \mathbf{Sp}^G$ is E-local if and only if $\Phi^H(X)$ is $\Phi^H(E)$-local for all $H \subset G$.*

2. *If $E \in \mathbf{Sp}$ is smashing, $X \in \mathbf{Sp}^G$ is i_*E-local if and only if $\Phi^H(X)$ is E-local for all $H \subset G$.*

3. *If $f : G \to G'$ is a group homomorphism, and $E \in \mathbf{Sp}^{G'}$ is smashing, $X \in \mathbf{Sp}^G$ is f^*E-local if and only if $\Phi^H(X)$ is $\Phi^{f(H)}(E)$-local for all $H \subset G$.*

4. *If $H \subset G$, and $E \in \mathbf{Sp}^H$ is smashing, $X \in \mathbf{Sp}^G$ is $N_H^G E$-local if and only if for all $K \subset G$, and for all $[g] \in K \backslash G / H$, $\Phi^K(X)$ is $\Phi^{K^g \cap H}(E)$-local.*

5. *If $E \in \mathbf{Sp}^G$ is smashing, and $\mathcal{Z}_{E^G} \subset \mathcal{Z}_{E^H}$ for all $H \subset G$ (e.g. if E is a ring spectrum), then $X \in \mathbf{Sp}^G$ is E^G-local if and only if i_*X is E-local.*

Proof. For (1), X is E-local iff the map $X \to L_E(X)$ is an equivalence, but this is true iff

$$\Phi^H(X) \to \Phi^H(L_E(X)) \simeq L_{\Phi^H(E)}(X)$$

is an equivalence for all H, i.e. $\Phi^H(X)$ is $\Phi^H(E)$-local for all H. The rest are immediate consequences of (1). □

The norm is unique among the above functors in that it does not in general preserve cofiber sequences. However, we have the following interesting corollary of 4.2.13:

Corollary 4.2.16. *If G is abelian, N_H^G preserves idempotent cofiber sequences. That is, if $e \to S^0 \to f \to \Sigma e$ is an idempotent triangle in $\mathbf{Sp}^H$, then $N_H^G(e) \to S^0 \to N_H^G(f)$ is a cofiber sequence in $\mathbf{Sp}^G$ such that*

$$N_H^G(e) \to S^0 \to N_H^G(f) \to \Sigma N_H^G(e)$$

is an idempotent triangle.

Proof. By 4.1.11, every idempotent triangle in $\mathbf{Sp}^H$ is of the form

$$Z_E(S^0) \to S^0 \to L_E(S^0) \to \Sigma Z_E(S^0)$$

We will show that the sequence

$$N_H^G(Z_E(S^0)) \to S^0 \to N_H^G(L_E(S^0))$$

is equivalent to the idempotent cofiber sequence

$$Z_{N_H^G E}(S^0) \to S^0 \to L_{N_H^G E}(S^0)$$

Note that since $N_H^G(*) = *$, $N_H^G(-)$ sends the zero map to the zero map. Therefore the composite $N_H^G(Z_E(S^0)) \to S^0 \to N_H^G(L_E(S^0))$ is null, and so we have a commutative diagram

$$\begin{tikzcd}
N_H^G(Z_E(S^0)) \arrow[r] \arrow[d, dashed, "f"] & S^0 \arrow[r] \arrow[d, "="] & N_H^G(L_E(S^0)) \arrow[d, "\simeq"] \\
Z_{N_H^G E}(S^0) \arrow[r] & S^0 \arrow[r] & L_{N_H^G E}(S^0)
\end{tikzcd}$$

To show that f is an equivalence, it suffices to show that $\Phi^K(f)$ is an equivalence for all $K \subset G$, hence it suffices to show $\Phi^K(-)$ of the top row is a cofiber sequence. This gives

$$\bigwedge_{[g] \in K\backslash G/H} \Phi^{K\cap H}(Z_E(S^0)) \to S^0 \to \bigwedge_{[g]\in K\backslash G/H} \Phi^{K\cap H}(L_E(S^0))$$

by 4.2.12, where we have used that G is abelian so that $K^g = K$. By 4.2.13, this may be further identified with

$$\bigwedge_{[g]\in K\backslash G/H} Z_{\Phi^{K\cap H}(E)}(S^0) \to S^0 \to \bigwedge_{[g]\in K\backslash G/H} L_{\Phi^{K\cap H}(E)}(S^0)$$

By 4.1.13, this is the idempotent cofiber sequence associated to $\mathcal{Z}_{\Phi^{K\cap H}(E)}$, as

$$\langle \Phi^{K\cap H}(E)\rangle = \langle \bigvee_{[g]\in K\backslash G/H} \Phi^{K\cap H}(E)\rangle = \langle \bigwedge_{[g]\in K\backslash G/H} \Phi^{K\cap H}(E)\rangle$$

since $\Phi^{K\cap H}(E)$ is smashing. □

Remark 4.2.17. It doesn't make sense to ask whether N_H^G preserves idempotent *triangles* in the sense of [6] because $N_H^G(S^1) \simeq S^{\mathrm{Ind}_H^G(1)}$, and so applying N_H^G to the idempotent triangle

$$e \to S^0 \to f \to \Sigma e$$

yields the sequence of maps

$$N_H^G(e) \to S^0 \to N_H^G(f) \to S^{\mathrm{Ind}_H^G(1)} \wedge N_H^G(e)$$

which is not a distinguished triangle in $\mathbf{Sp}^G$ unless $H = G$ or $e \simeq *$. We have only shown that, when G is abelian,

$$N_H^G(e) \to S^0 \to N_H^G(f)$$

is a cofiber sequence, and in particular the first two morphisms in an idempotent triangle.

We now give a counterexample to the above claim in the general case when G is not necessarily abelian.

Proposition 4.2.18. *Fix an inclusion $C_2 \hookrightarrow \Sigma_3$. The corresponding functor $N_{C_2}^{\Sigma_3} : \mathbf{Sp}^{C_2} \to \mathbf{Sp}^{\Sigma_3}$ does not preserve all idempotent cofiber sequences.*

Proof. Consider the idempotent cofiber sequence

$$EC_{2+} \to S^0 \to \tilde{E}C_2$$

in $\mathbf{Sp}^{C_2}$. Applying $N_{C_2}^{\Sigma_3}$ yields the sequence

$$E\mathcal{F}_{C_3+} \to S^0 \to \tilde{E}\mathcal{P}$$

which is not a cofiber sequence. □

4.2.3 Induction and smashing localizations

By far the most interesting change of group functor with respect to smashing localizations is induction, because it is not monoidal, and hence we treat it separately. We find that induced G-spectra $G_+ \wedge_H E$ are rarely smashing, though we give a necessary and sufficient condition for $G_+ \wedge_H E$ to be smashing.

Proposition 4.2.19. *Suppose $H \subset G$, and $E \in \mathbf{Sp}^H$. Then $G_+ \wedge_H E$ is smashing if and only if $\Phi^K(L_{G_+\wedge_H E}(S^0)) \simeq *$ for all $K \notin \mathcal{F}_H$ and $i_H^G(G_+ \wedge_H E)$ is smashing.*

Proof. Suppose $G_+ \wedge_H E$ is smashing, then any restriction of it is also smashing. If $K \notin \mathcal{F}_H$, let $\mathcal{F}$ be the smallest family containing H and every proper subgroup of K. It follows that $\tilde{E}\mathcal{F} \in \mathcal{Z}_{G/H_+} \subset \mathcal{Z}_{G_+\wedge_H E}$, as $i_H^G \tilde{E}\mathcal{F} \simeq *$, hence

$$\Phi^K(G_+ \wedge_H E) \simeq *$$

Since $G_+ \wedge_H E$ is smashing, $\mathcal{Z}_{G_+\wedge_H E} = \mathcal{Z}_{L_{G_+\wedge_H E}(S^0)}$, so that

$$\Phi^K(L_{G_+\wedge_H E}(S^0)) \simeq *$$

Conversely, suppose $\Phi^K(L_{G_+\wedge_H E}(S^0)) \simeq *$ for all $K \notin \mathcal{F}_H$ and $i_H^G(G_+ \wedge_H E)$ is smashing. To show $L_{G_+\wedge_H E}$ is smashing, it suffices to show then that

$$\langle \Phi^K(G_+ \wedge_H E) \rangle = \langle \Phi^K(L_{G_+\wedge_H E}(S^0)) \rangle$$

for all $K \in \mathcal{F}_H$. This may be checked after restriction to H, where it follows immediately from the fact that that $i_H^G(G_+ \wedge_H E)$ is smashing, using 4.2.2. □

Corollary 4.2.20. *If $H \subset G$ is normal, and $E \in \mathbf{Sp}^H$ is smashing, then $G_+ \wedge_H E$ is smashing if and only if $\Phi^K(L_{G_+\wedge_H E}(S^0)) \simeq *$ for all $K \subset H$.*

Proof. This follows immediately from the previous proposition along with the observation that

$$\langle i_H^G(G_+ \wedge_H E)\rangle = \langle \bigvee_{[g]\in G/H} {}^gE\rangle = \bigvee_{[g]\in G/H} \langle {}^gE\rangle$$

is a smashing Bousfield class by 4.1.13 since ${}^gE$ is smashing for all g. □

When $H \subset G$ is normal, we arrive at a somewhat explicit formula for an induced localization, which we can interpret as follows: induced smashing localizations are smashing *after* H-cofree completion. When $H = \{e\}$, this can be further related to the corresponding trivial localization.

Proposition 4.2.21. *If $H \subset G$ is normal, $E \in \mathbf{Sp}^H$ is smashing, and $X \in \mathbf{Sp}^G$, then*

$$L_{G_+\wedge_H E}(X) \simeq L_{G/H_+}(L_{G_+\wedge_H E}(S^0) \wedge X) \simeq F(E\mathcal{F}_{H+}, L_{G_+\wedge_H E}(S^0) \wedge X)$$

Proof. The map

$$X \to F(E\mathcal{F}_{H+}, L_{G_+\wedge_H E}(S^0) \wedge X)$$

is a $G_+ \wedge_H E$ equivalence since $i_H^G(G_+ \wedge_H E)$ is smashing, and the target is easily seen to be $G_+ \wedge_H E$-local from 4.2.7. □

Proposition 4.2.22. *Let $E \in \mathbf{Sp}$ be any spectrum, and $X \in \mathbf{Sp}^G$, then*

$$L_{G_+\wedge E}(X) \simeq F(EG_+, i_* L_E X)$$

Proof. The map $X \to F(EG_+, L_{i_*E}X)$ becomes an equivalence after smashing with $G_+ \wedge E$, and the target is $G_+ \wedge E$-local by 4.2.7. □

Corollary 4.2.23. *Let $E \in \mathbf{Sp}$ be a smashing spectrum, then $G_+ \wedge E$ is smashing if and only if $(L_E(S^0))^{tH} \simeq *$ for all nontrivial subgroups $H \subset G$.*

Proof. $G_+ \wedge E$ is smashing if $\Phi^H(L_{G_+\wedge E}(S^0)) \simeq *$ for all nontrivial subgroups H, but

$$\begin{aligned}\Phi^H(L_{G_+\wedge E}(S^0)) &\simeq \Phi^H(F(EG_+, i_* L_E(S^0))) \\ &\simeq \Phi^H(F(EH_+, i_* L_E(S^0)))\end{aligned}$$

which is a module over the ring $(L_E(S^0))^{tH}$. Conversely, if $G_+ \wedge E$ is smashing, then $\mathcal{Z}_{G_+\wedge E} = \mathcal{Z}_{F(EG_+, i_* L_E(S^0))}$, but $\tilde{E}G \wedge G_+ \wedge E \simeq *$, and

$$(L_E(S^0))^{tH} \simeq (\tilde{E}G \wedge F(EG_+, i_* L_E(S^0)))^H$$

□

Corollary 4.2.24. *Let $E = E(n)$ at the prime p, then for all G such that p divides $|G|$, $G_+ \wedge E$ is smashing if and only if $n = 0$.*

Proof. When $n = 0$, $E(0) = H\mathbb{Q} = L_0(S^0)$, and $H\mathbb{Q}^{tH} \simeq *$ for all H nontrivial. If $n > 0$, then G has an element of order p and hence if $G_+ \wedge E$ were smashing, we would necessarily have $(L_n(S^0))^{tC_p} \simeq *$. However, we know from the main result of [47] that this Tate spectrum is not contractible. Indeed, they show that

$$\langle (L_n(S^0))^{tC_p} \rangle = \langle L_{n-1}(S^0) \rangle$$

hence the result follows from the fact that $L_{n-1}(S^0)$ is not trivial. □

We end this section with an example illustrating the necessity of the normality conditions in 4.2.20 and 4.2.21. It shows that if $E \in \mathbf{Sp}^H$, then E and $i_H^G(G_+ \wedge_H E)$ are not always Bousfield equivalent, and E being smashing does not always guarantee that $i_H^G(G_+ \wedge_H E)$ is smashing.

Proposition 4.2.25. *Let $G = \Sigma_4$ and $H = D_8 = \langle (1234), (13) \rangle \subset \Sigma_4$. Then $\tilde{E}\mathcal{F}_{\langle (1234) \rangle}$ is a smashing D_8-spectrum, but $i_{D_8}^{\Sigma_4}(\Sigma_{4+} \wedge_{D_8} \tilde{E}\mathcal{F}_{\langle (1234) \rangle})$ is not smashing.*

Proof. $\tilde{E}\mathcal{F}_{\langle(1234)\rangle}$ is smashing by 4.1.12. We have

$$D_8 = \{e, (13)(24), (12)(34), (14)(23), (1234), (1432), (13), (24)\} \subset \Sigma_4$$

$$D_8 \backslash \Sigma_4 / D_8 = \{D_8, D_8(12)D_8\}$$

$${}^{(12)}D_8 = \{e, (13)(24), (12)(34), (14)(23), (1342), (1243), (14), (23)\}$$

$$D_8 \cap {}^{(12)}D_8 = \{e, (13)(24), (12)(34), (14)(23)\} = V_4$$

Therefore we have

$$i^{\Sigma_4}_{D_8} \Sigma_{4+} \wedge_{D_8} \tilde{E}\mathcal{F}_{\langle(1234)\rangle} = \tilde{E}\mathcal{F}_{\langle(1234)\rangle} \vee \left(D_{8+} \wedge_{V_4} i^{{}^{(12)}D_8}_{V_4} \left({}^{(12)}\tilde{E}\mathcal{F}_{\langle(1234)\rangle} \right) \right)$$

${}^{(12)}\tilde{E}\mathcal{F}_{\langle(1234)\rangle}$ is the universal ${}^{(12)}D_8$ space $\tilde{E}\mathcal{F}_{\langle(1342)\rangle}$, and hence $i^{{}^{(12)}D_8}_{V_4}\left({}^{(12)}\tilde{E}\mathcal{F}_{\langle(1234)\rangle} \right)$ is the universal V_4-space $\tilde{E}\mathcal{F}_{\langle(14)(23)\rangle}$. Therefore we may write

$$i^{\Sigma_4}_{D_8} \Sigma_{4+} \wedge_{D_8} \tilde{E}\mathcal{F}_{\langle(1234)\rangle} = \tilde{E}\mathcal{F}_{\langle(1234)\rangle} \vee \left(D_{8+} \wedge_{V_4} \tilde{E}\mathcal{F}_{\langle(14)(23)\rangle} \right)$$

We now assume for the sake of contradiction that this D_8-spectrum is smashing, hence we restrict to $\langle(1234)\rangle \cong C_4$ to get a smashing C_4-spectrum

$$i^{D_8}_{\langle(1234)\rangle} \left(\tilde{E}\mathcal{F}_{\langle(1234)\rangle} \vee \left(D_{8+} \wedge_{V_4} \tilde{E}\mathcal{F}_{\langle(14)(23)\rangle} \right) \right) \simeq i^{D_8}_{\langle(1234)\rangle} \left(D_{8+} \wedge_{V_4} \tilde{E}\mathcal{F}_{\langle(14)(23)\rangle} \right)$$

One checks that

$$\langle(1234)\rangle \backslash D_8 / V_4 = \{\langle(1234)\rangle e V_4\}$$

$$\langle(1234)\rangle \cap V_4 = \langle(13)(24)\rangle$$

so that

$$\begin{aligned} i^{D_8}_{\langle(1234)\rangle} \left(D_{8+} \wedge_{V_4} \tilde{E}\mathcal{F}_{\langle(14)(23)\rangle} \right) &\simeq \langle(1234)\rangle_+ \wedge_{V_4 \cap \langle(1234)\rangle} i^{V_4}_{V_4 \cap \langle(1234)\rangle} \tilde{E}\mathcal{F}_{\langle(14)(23)\rangle} \\ &\simeq \langle(1234)\rangle_+ \wedge_{\langle(13)(24)\rangle} \tilde{E}\mathcal{F}_{\langle(14)(23)\rangle \cap \langle(13)(24)\rangle} \\ &\simeq C_{4+} \wedge_{C_2} \tilde{E}C_2 \end{aligned}$$

Now $\tilde{E}C_2$ is a smashing C_2-spectrum, and so by 4.2.20, $C_{4+} \wedge_{C_2} \tilde{E}C_2$ is a smashing C_4-spectrum if and only if

$$\Phi^{C_4}(L_{C_{4+}\wedge_{C_2}\tilde{E}C_2}(S^0)) \simeq *$$

and the proof of 4.2.21 shows that $L_{C_{4+}\wedge_{C_2}\tilde{E}C_2}(S^0) = F(E\mathcal{P}_+, \tilde{E}C_4)$, but

$$\Phi^{C_4}(F(E\mathcal{P}_+, \tilde{E}C_4)) \simeq (S^0)^{tC_2} \not\simeq *$$

□

4.3 Consequences for chromatic localizations

4.3.1 C_{2^n}-Borel Ravenel conjectures

In this section, we prove the main theorems stated in the beginning of this chapter, namely that the analogs of the smash product theorem and the chromatic convergence theorem for the $E_{\mathbb{R}}(n)$'s hold only after cofree completion. We remark on analogs of the nilpotence and thick subcategory theorems. We also discuss recent C_{2^n}-equivariant analogs of the $E_{\mathbb{R}}(n)$'s constructed in [11], and we identify their Bousfield classes.

Theorem 4.3.1. *If $n > 0$, then $E_{\mathbb{R}}(n)$ is not smashing. Moreover, for $X \in \mathbf{Sp}^{C_2}$,*

$$L_{E_{\mathbb{R}}(n)}(X) \simeq F(EC_{2+}, i_* L_{E(n)}(S^0) \wedge X)$$

Proof. By 4.1.4, we have

$$\langle E_{\mathbb{R}}(n)\rangle = \langle C_{2+} \wedge E(n)\rangle$$

and now the result follows from 4.2.22 and 4.2.24. For the claim about $\Phi^{C_2}(E_{\mathbb{R}}(n))$, $E_{\mathbb{R}}(n)$ is a module over $MU_{\mathbb{R}}[\overline{v_n}^{-1}]$, and $\Phi^{C_2}(MU_{\mathbb{R}}[\overline{v_n}^{-1}]) \simeq *$ as $\Phi^{C_2}(\overline{v_n}) = 0$ ([39], 5.50). □

Theorem 4.3.2. *If X is a 2-local finite C_2-spectrum, we have a diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & F(EC_{2+}, X) \arrow[d, "\simeq"] \\ & \varprojlim_n L_{E_{\mathbb{R}}(n)}(X) \end{tikzcd}$$

Proof. The category of cofree C_2-spectra is closed under homotopy limits, hence there exists a unique up to homotopy vertical map making the above diagram commute. As a map between cofree C_2-spectra, it is an equivalence if and only if it induces an underlying equivalence. The underlying map is an equivalence by the nonequivariant chromatic convergence theorem (see [79]). □

Remark 4.3.3. The $MU_{\mathbb{R}}$ analogs of the nilpotence and thick subcategory theorems also fail in genuine C_2-spectra, and this is much easier to see. For the nilpotence theorem, the class $2 - [C_2] \in \pi_0^{C_2}(S^0) \cong A(C_2)$ goes to 0 in $\pi_0^{C_2}(MU_{\mathbb{R}}) = \mathbb{Z}$, but it is not nilpotent in $A(C_2)$. Passing to Borel C_2-spectra does not correct this: the endomorphism ring of the unit in Borel C_2-spectra is $A(C_2)^{\wedge}_I$, by Lin's theorem [56], and $2 - [C_2]$ is still not nilpotent.

In the case of the thick subcategory theorem, the Balmer spectrum of $(\mathbf{Sp}^{C_2})^{\omega}$ was determined in [7], and as remarked in the introduction, for $n > 0$, the collection of finite acyclics of $E_{\mathbb{R}}(n)$ do not determine all of the thick tensor ideals in $(\mathbf{Sp}^{C_2})^{\omega}$, so no reasonable analog of the thick subcategory theorem for the $E_{\mathbb{R}}(n)$'s (or the $K_{\mathbb{R}}(n)$'s) can hold in $\mathbf{Sp}^{C_2}$. Passing to Borel, we run into the following issue: the unit is not compact in Borel C_2-spectra, and in particular the compact objects and dualizable objects do not coincide. This makes an analysis of the spectrum more difficult, but is a subject we plan to revisit in future work.

In [11], Beaudry, Hill, Shi, and Zeng construct genuine C_{2^n}-spectra that serve as analogs to the $E_{\mathbb{R}}(n)$'s. We recall their construction, which hinges on the observation that one may construct the C_{2^n}-spectrum

$$BP^{((G))}\langle m\rangle := N_{C_2}^{C_{2^n}} BP_{\mathbb{R}}/(C_{2^n}\cdot \overline{t_{m+1}}, C_{2^n}\cdot \overline{t_{m+2}}, \ldots)$$

and for a carefully chosen class $D \in \pi^G_{*\rho_G} BP^{((G))}$, $D^{-1}BP^{((G))}\langle m\rangle$ should have height $h = 2^{n-1}m$. More precisely, they show:

Theorem 4.3.4. *[11, Theorems 1.5 and 1.8] For $h = 2^{n-1}m$, there is a class $D \in \pi^G_{*\rho_G} BP^{((G))}$ and a height h formal group law Γ_h over $\mathbb{F}_2$ such that for any perfect field k of characteristic*

2, if we regard the corresponding Lubin-Tate theory $E(k,\Gamma_h)$ as a cofree C_{2^n}-spectrum, there is a diagram in $\mathbf{Sp}^{C_{2^n}}$

$$\begin{tikzcd} BP^{((G))} \arrow[r] \arrow[d] & E(k,\Gamma_h) \\ D^{-1}BP^{((G))} \arrow[ur, dashed] & \end{tikzcd}$$

It follows that the above map factors further through

$$E_G(m) := D^{-1}BP^{((G))}\langle m\rangle$$

which can be thought of as a C_{2^n}-equivariant height h Johnson-Wilson theory. The corresponding localization functors on $\mathbf{Sp}^{C_{2^n}}$ behave formally very similarly to those of the $E_{\mathbb{R}}(n)$'s, as we can identify their Bousfield classes in a similar way. We need the following results about the class D:

Theorem 4.3.5. *[11, Theorem 1.2] The element*

$$i_e^G D \in \pi_*^e BP^{((G))}\langle m\rangle \cong \mathbb{Z}_{(2)}[G\cdot \overline{t_1},\dots,G\cdot\overline{t_m}]$$

satisfies the following properties:

- *v_h divides D,*
- *$(2,v_1,\dots,v_h)$ is a regular sequence in $D^{-1}\pi_*^e BP^{((G))}\langle m\rangle$,*
- *$v_r \in I_r$ for $r > h$,*
- *$D^{-1}\pi_*^e(BP^{((G))}\langle m\rangle)/I_h \cong \mathbb{F}_2[(t_m^G)^{\pm}]$ with $v_h = t_m^{(2^h-1)/(2^m-1)}$, and*
- *the formal group law carried by $\pi_*^e BP^{((G))}\langle m\rangle$ has height exactly h over the ring*

$$D^{-1}\pi_*^e(BP^{((G))}\langle m\rangle)/I_h$$

Corollary 4.3.6. *The underlying spectrum of $E_G(m)$ is Bousfield equivalent to $E(h)$, and the geometric fixed points of $E_G(m)$ at any nontrivial subgroup is contractible. In particular,*

$$\langle E_G(m)\rangle = \langle C_{2^n+} \wedge E(h)\rangle$$

Proof. For the claim about geometric fixed points, [11, Theorem 1.8] show that the class D is divisible by norms of certain classes from $\pi^{C_2}_{*\rho_{C_2}} BP^{((G))}$, all of which become null upon applying Φ^H for any nontrivial subgroup $H \subset C_{2^n}$, and the result follows as in [39, Section 10].

For the underlying spectrum, the conditions in 4.3.5 are enough to guarantee that the map

$$\mathrm{Spec}(D^{-1}\pi^e_* BP^{((G))}\langle m\rangle) \to \mathcal{M}_{FG}$$

factors through a faithfully flat cover of the open substack $\mathcal{M}^{\leq h}_{FG}$, and any such Landweber theory is Bousfield equivalent to $E(h)$ by Remark 2.2.22. In more detail, items (1) and (2) in 4.3.5 guarantee that the spectrum $i^{C_{2^n}}_e(D^{-1}BP^{((G))}\langle m\rangle)$ is Landweber exact, and by functoriality it maps to the Landweber exact spectrum E with coefficient ring

$$E_* := (D^{-1}\pi^e_* BP^{((G))}\langle m\rangle)[u]/(u^{2^m-1} - t_m)$$

with $|u| = 2$, which is 2-periodic. $(D^{-1}\pi^e_* BP^{((G))}\langle m\rangle)[u]/(u^{2^m-1} - t_m)$ is a free module over $D^{-1}\pi^e_* BP^{((G))}\langle m\rangle$, so the inclusion is faithfully flat, and the two Landweber theories are Bousfield equivalent. In E_*, we may use u to conjugate the formal group into degree 0, and now a spectrum X is E-acyclic if and only if the corresponding quasicoherent sheaves on $\mathcal{M}_{FG}$ determined by $E_0(X)$ and $E_1(X)$ are zero. It now suffices to show that the map

$$\mathrm{Spec}(E_0) \to \mathcal{M}^{\leq h}_{FG}$$

is a flat cover.

This map is flat by the Landweber exact functor theorem, so it suffices to show that it is essentially surjective, and by 4.3.4, there is a factorization

$$\begin{tikzcd} \mathrm{Spec}(E(k,\Gamma_h)_0) \arrow[r] \arrow[dr, "p"'] & \mathrm{Spec}(E_0) \arrow[d] \\ & \mathcal{M}^{\leq h}_{FG} \end{tikzcd}$$

and p is a faithfully flat cover, as $\mathrm{Spec}(E(k,\Gamma_h)_0)$ is a Lubin-Tate universal space of height h. ∎

The Bousfield classes of the $E_G(m)$'s are therefore nested: we see that

$$\mathcal{Z}_{E_G(m)} \subset \mathcal{Z}_{E_G(m-1)}$$

and hence for any $X \in \mathbf{Sp}^{C_{2^n}}$, we may form a chromatic tower

$$X \to \cdots \to L_{E_G(m)}(X) \to L_{E_G(m-1)}(X) \to \cdots \to L_{E_G(0)}(X)$$

Our results for C_2 now follow in essentially the same way for the $E_G(m)$'s:

Theorem 4.3.7. *Let $E_G(m)$ denote the C_{2^n}-spectrum $D^{-1}BP^{((G))}\langle m\rangle$ constructed in [11], where $h = 2^{n-1}m$.*

- *If $m > 0$, then $E_G(m)$ is not smashing. Moreover, for $X \in \mathbf{Sp}^{C_{2^n}}$,*

$$L_{E_G(m)}(X) \simeq F(EC_{2^n +}, i_* L_{E(h)}(S^0) \wedge X)$$

- *If X is a 2-local finite C_{2^n}-spectrum, we have a diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & F(EC_{2^n +}, X) \arrow[d, "\simeq"] \\ & \varprojlim_m L_{E_G(m)}(X) \end{tikzcd}$$

4.3.2 Smashing C_{p^n}-spectra

In light of 4.3.1, a natural question from here is then if the $E_{\mathbb{R}}(n)$ are not smashing, can we construct equivariant spectra analogous to the $E(n)$ that *are* smashing? More specifically, every thick tensor ideal in $\mathbf{Sp}^{\omega}_{(p)}$ is the collection of finite acyclics of one of the $E(n)$'s, so we may ask if a similar statement is true for $(\mathbf{Sp}^G)^{\omega}_{(p)}$, and we can give a construction when $G = C_{p^n}$. The following theorem was proven in the case $n = 1$ by Balmer and Sanders [7], and for $n > 1$ by Barthel, Hausmann, Naumann, Nikolaus, Noel, and Stapleton [9].

Theorem 4.3.8. *[7] [9] The thick tensor ideals in $(\mathbf{Sp}^{C_{p^n}}_{(p)})^c$ are precisely the subcategories of the form*

$$\{X \in (\mathbf{Sp}^{C_{p^n}}_{(p)})^c : \Phi^{C_{p^i}}(X) \in \mathcal{Z}_{E(m_i)}\}$$

where $m_i \leq m_{i+j} + 1$ for all $0 \leq i \leq n-1$ and $1 \leq j \leq n-i$.

Therefore for $G = C_{p^n}$, the above question becomes: for a sequence of natural numbers $m_0, \dots, m_n$ with $m_i \le m_{i+j} + 1$ for all $0 \le i \le n-1$ and $1 \le j \le n-i$, can we build a *smashing* G-spectrum $E(m_0, \dots, m_n)$ so that $\Phi^{C_{p^i}}(E(m_0, \dots, m_n)) \simeq E(m_i)$? It is easy to build $E(m_0, \dots, m_n)$ with the stated geometric fixed points since we may assume by induction that $E(m_1, \dots, m_n) \in \mathbf{Sp}^{C_{p^{n-1}}}$ exists, and then set

$$E(m_0, \dots, m_n) := (E{C_{p^n}}_+ \wedge i_* E(m_0)) \vee (\tilde{E}C_{p^n} \wedge q^* E(m_1, \dots, m_n))$$

where $q : C_{p^n} \to C_{p^{n-1}}$ is the quotient map. It is not obvious that this spectrum is smashing, but using the results of Section 4.2, we can build a different representative of the same Bousfield class that is manifestly smashing.

We do not know if there is a way to construct such spectra that are not split as above. However, what follows would show that any such construction produces a smashing G-spectrum, since it would be Bousfield equivalent to the ones we construct. We begin with the case $n = 1$.

Proposition 4.3.9. *For every pair of natural numbers m_0, m_1, there is a smashing C_p-spectrum $E(m_0, m_1)$ with the property that*

$$\langle \Phi^{C_{p^i}}(E(m_0, m_1)) \rangle = \langle E(m_i) \rangle$$

for $i = 0, 1$ if and only if $m_0 \le m_1 + 1$.

Proof. Setting $E(m_0, m_1) = (C_{p_+} \wedge E(m_0)) \vee (\tilde{E}C_p \wedge i_* E(m_1))$, one checks easily the claim about Bousfield classes, and thus any model for $E(m_0, m_1)$ is Bousfield equivalent to this one. $E(m_0, m_1)$ is smashing if and only if, for any family $\{Y_i\}$ of $E(m_0, m_1)$-locals, the map

$$\phi : \bigvee_i Y_i \to L_{m_0, m_1}\left(\bigvee_i Y_i \right)$$

is an equivalence. It always induces an underlying equivalence as $i_e^{C_p} \circ L_{m_0, m_1} \simeq L_{m_0} \circ i_e^{C_p}$, so $E(m_0, m_1)$ is smashing if and only if $\Phi^{C_p}(\phi)$ is an equivalence. We claim this is true if and only if $\Phi^{C_p}(L_{m_0,m_1}(S^0))$ is $E(m_1)$-local. If $\Phi^{C_p}(L_{m_0,m_1}(S^0))$ were $E(m_1)$-local, then $\Phi^{C_p}(L_{m_0,m_1}(Y))$ would be $E(m_1)$-local for any Y, as a module over an $E(m_1)$-local ring spectrum. But then $\Phi^{C_p}(\phi)$ would be an $E(m_1)$-equivalence between $E(m_1)$-locals. Conversely, if $\Phi^{C_p}(L_{m_0,m_1}(S^0))$ were not $E(m_1)$-local, then we could not have

$$\langle \Phi^{C_p}(L_{m_0,m_1}(S^0)\rangle = \langle \Phi^{C_p}(E(m_0,m_1))\rangle$$

as $\langle \Phi^{C_p}(E(m_0,m_1))\rangle = \langle E(m_1)\rangle$.

From Section 4.2, we have

$$L_{C_{p+}\wedge E(m_0)}(X) \simeq F(EC_{p+}, i_* L_{m_0}(S^0)\wedge X)$$
$$L_{\tilde{E}C_p\wedge i_* E(m_1)}(X) \simeq \tilde{E}C_p \wedge i_* L_{m_1}(S^0)\wedge X$$

It follows that $L_{C_{p+}\wedge E(m_0)} \circ L_{\tilde{E}C_p\wedge i_* E(m_1)} \simeq *$, and hence by a general argument (see Bauer in [22]), there is a natural homotopy pullback square

$$\begin{tikzcd}
L_{m_0,m_1}(X) \arrow[rr] \arrow[d] & & F(EC_{p+}, i_* L_{m_0}(S^0)\wedge X) \arrow[d] \\
\tilde{E}C_p \wedge i_* L_{m_1}(S^0)\wedge X \arrow[r] & \tilde{E}C_p \wedge i_* L_{m_1}(S^0)\wedge F(EC_{p+}, i_* L_{m_0}(S^0)\wedge X) &
\end{tikzcd}$$

Setting $X = S^0$, and applying $\Phi^{C_p}(-)$, we have a homotopy pullback square

$$\begin{tikzcd}
\Phi^{C_p}(L_{m_0,m_1}(S^0)) \arrow[r] \arrow[d] & L_{m_0}(S^0)^{tC_p} \arrow[d] \\
L_{m_1}(S^0) \arrow[r] & L_{m_1}(L_{m_0}(S^0)^{tC_p})
\end{tikzcd}$$

and by the main result of [47], the right hand map is an equivalence if and only if $m_0 \leq m_1 + 1$. $\square$

Theorem 4.3.10. *For every sequence of natural numbers $m_0,\ldots,m_n$ satisfying $m_i \leq m_{i+j}+1$ for all $0 \leq i \leq n-1$ and $1 \leq j \leq n-i$, there is a smashing C_{p^n}-spectrum $E(m_0,\ldots,m_n)$ with the property that*

$$\Phi^{C_{p^i}}(E(m_0,\ldots,m_n)) \simeq E(m_i)$$

for all $0 \leq i \leq n$.

Proof. We proceed by induction on n, and we may assume $n > 1$ by the previous proposition. As stated above, it suffices to show there is a spectrum $E(m_0, \ldots, m_n)$ with the property that

$$\langle \Phi^{C_{p^i}}(E(m_0, \ldots, m_n)) \rangle = \langle E(m_i) \rangle$$

for all i. There are 3 cases to check:

(i) $m_0 = m_1$: By induction, we may assume there is a smashing $C_{p^{n-1}}$-spectrum $E(m_1, \ldots, m_n)$ with the stated properties. Let $q : C_{p^n} \to C_{p^{n-1}}$ be the usual quotient map. Then $E(m_0, \ldots, m_n) := q^* E(m_1, \ldots, m_n)$ is a smashing C_{p^n}-spectrum and

$$\Phi^{C_{p^i}}(q^* E(m_1, \ldots, m_n)) = \begin{cases} \Phi^{C_{p^{i-1}}}(E(m_1, \ldots, m_n)) & i > 0 \\ \Phi^{\{e\}}(E(m_1, \ldots, m_n)) & i = 0 \end{cases}$$

(ii) $m_0 < m_1$. Here we set

$$E(m_0, \ldots, m_n) := i_* E(m_0) \vee (\tilde{E}C_{p^n} \wedge q^* E(m_1, \ldots, m_n))$$

This is a smashing C_{p^n}-spectrum as in 4.1.13, and we have

$$\Phi^{C_{p^i}}(E(m_0, \ldots, m_n)) = \begin{cases} E(m_0) \vee \Phi^{C_{p^{i-1}}}(E(m_1, \ldots, m_n)) & i > 0 \\ E(m_0) & i = 0 \end{cases}$$

Note however that $m_0 \le m_i$ for all $i > 0$ as $m_i \ge m_1 - 1$ for all $i > 1$, hence $\langle E(m_0) \vee E(m_i) \rangle = \langle E(m_i) \rangle$.

(iii) $m_0 = m_1 + 1$. Since we have assumed $n > 1$, we can form the smashing C_{p^n}-spectrum

$$E(m_0, \ldots, m_n) := N_{C_p}^{C_{p^{n+1}}} E(m_0, m_1) \vee (\tilde{E}C_{p^n} \wedge q^* E(m_1, \ldots, m_n))$$

and we have

$$\Phi^{C_{p^i}}(E(m_0, \ldots, m_n)) = \begin{cases} \Phi^{C_p}(E(m_0, m_1))^{\wedge k(i)} \vee \Phi^{C_{p^{i-1}}}(E(m_1, \ldots, m_n)) & i > 0 \\ E(m_0)^{\wedge p^{n-1}} & i = 0 \end{cases}$$

where $k(i)$ is some positive integer that won't affect the Bousfield class. Note that since $m_0 = m_1 + 1$, $m_i \geq m_1$ for all $i > 0$, hence we have

$$\langle \Phi^{C_{p^i}}(E(m_0, \ldots, m_n)) \rangle = \langle E(m_1) \rangle \vee \langle E(m_i) \rangle = \langle E(m_i) \rangle$$

for $i > 0$, and

$$\langle \Phi^{\{e\}}(E(m_0, \ldots, m_n)) \rangle = \langle E(m_0)^{\wedge p^{n-1}} \rangle = \langle E(m_0) \rangle$$

□

4.4 Consequences for localizations of N_∞-algebras

In this section, we collect some results on Bousfield localizations of N_∞-algebras, and we refer the reader to [12] for the basic definitions in the theory of N_∞-algebras. We begin this section by recalling a surprising theorem of Blumberg and Hill. In this section, $\mathrm{Map}_G(-,-)$ will denote the G-space of maps in the category of G-spaces.

Theorem 4.4.1. *[12, Theorem 1.4] If $\mathcal{O}$ is an N_∞-operad, and R is an $\mathcal{O}$-algebra such that R is cofree, then R is equivalent (as an $\mathcal{O}$-algebra) to a genuine G-E_∞ ring.*

Proof. $R \simeq F(EG_+, R)$, and since R is an $\mathcal{O}$-algebra, $F(EG_+, R)$ is canonically a $\mathrm{Map}_G(EG, \mathcal{O})$-algebra. Each $\mathrm{Map}_G(EG, \mathcal{O}_n)$ is a universal space for some family $\mathcal{F}$ of graph subgroups of $G \times \Sigma_n$, and if $\mathcal{F}'$ is any other such family, a map

$$E\mathcal{F}' \to \mathrm{Map}_G(EG, \mathcal{O}_n)$$

is the same thing as a map

$$E\mathcal{F}' \times EG \to \mathcal{O}_n$$

But $E\mathcal{F}' \times EG \simeq EG$, hence there is always such a map, as EG is initial in the category of such universal spaces. Therefore $\mathrm{Map}_G(EG, \mathcal{O}_n)$ is terminal, so it is an $E_G\Sigma_n$, and $\mathrm{Map}_G(EG, \mathcal{O})$ is equivalent to the terminal N_∞-operad. □

This is an extremely useful theorem: many genuine equivariant homotopy types come naturally as cofree spectra equipped with naive E_∞-structures. The Morava E theories E_n with their actions by (subgroups of) the Morava stabilizer group, furnished by the Goerss-Hopkins-Miller theorem [82], come to us this way, and similarly for various equivariant forms of TMF. For example, the C_2-spectrum $Tmf_1(3)$ studied by Hill and Meier [40]. These cofree theories E therefore come equipped with canonical maps of genuine commutative ring spectra

$$N^T E \to E$$

for finite G-sets T. These maps play an essential role in computations involving the above spectra, see for example [34, Section 6] in the E_n-case.

We give a series of generalizations of this result that concern H-cofree G-spectra and induced localizations. This is a natural direction of generalization as $F(EG_+, -)$ is simply the induced Bousfield localization functor $L_{G_+}(-)$. Let $E \in \mathbf{Sp}^G$ and let $\underline{\mathcal{Z}_E}$ denote the nonunital symmetric monoidal coefficient system (as in [38]) of E-acyclics. That is, $\underline{\mathcal{Z}_E}$ is the contravariant (pseudo-)functor from the orbit category $\mathcal{O}_G$ to nonunital symmetric monoidal categories with values

$$\underline{\mathcal{Z}_E}(G/H) = \mathcal{Z}_{i_H^G E}$$

We now recall the following theorem of Hill-Hopkins and Gutierrez-White:

Theorem 4.4.2. *[38] [33] Let $\mathcal{O}$ be an N_∞-operad for the group G, and $E \in \mathbf{Sp}^G$. Then $L_E(-)$ preserves $\mathcal{O}$-algebras if and only if for all $K \subset H \subset G$ such that H/K is an admissible H-set of $\mathcal{O}$,*

$$N^{H/K}(\underline{\mathcal{Z}_E}(G/H)) \subset \underline{\mathcal{Z}_E}(G/H)$$

Proposition 4.4.3. *Let $\mathcal{O}$ be an N_∞ operad for the group G. If $H \subset G$ and $E \in \mathbf{Sp}^H$ is such that $L_E(-)$ preserves $i_H^G\mathcal{O}$-algebras, then $L_{G_+ \wedge_H E}(-)$ preserves $\mathcal{O}$-algebras.*

Proof. Let $K' \subset K \subset G$ be such that K/K' is an admissible K-set for $\mathcal{O}$, then we must show that

$$N^{K/K'}\underline{\mathcal{Z}_{G_+\wedge_H E}}(G/K) \subset \underline{\mathcal{Z}_{G_+\wedge_H E}}(G/K)$$

The double coset formula states

$$i_K^G G_+ \wedge_H E = \bigvee_{[g]\in K\backslash G/H} K_+ \wedge_{K\cap^g H} i_{K\cap^g H}^{^gH}(^gE)$$

hence $Z \in \underline{\mathcal{Z}_{G_+\wedge_H E}}(G/K) \iff i_{K\cap^g H}^K Z \in \mathcal{Z}_{i_{K\cap^g H}^{^gH}(^gE)}$ for all $g \in G$. We therefore assume that $i_{K\cap^g H}^K Z \in \mathcal{Z}_{i_{K\cap^g H}^{^gH}(^gE)}$, and we must show that $i_{K\cap^g H}^K N^{K/K'}(Z) \in \mathcal{Z}_{i_{K\cap^g H}^{^gH}(^gE)}$, but we have

$$i_{K\cap^g H}^K N^{K/K'}(Z) = \bigwedge_{[h]\in(K\cap^g H)\backslash K/K'} N_{(K\cap^g H)\cap^h(K')}^{K\cap^g H} i_{(K\cap^g H)\cap^h(K')}^{^h(K')}(^h(i_{K'}^K Z))$$

This smash product is in $\mathcal{Z}_{i_{K\cap^g H}^{^gH}(^gE)}$ if any of its factors is, hence we may take $h = e$ so that it suffices to show that

$$N_{(K\cap^g H)\cap K'}^{K\cap^g H} i_{(K\cap^g H)\cap K'}^{K'}(i_{K'}^K Z) = N^{(K\cap^g H)/((K\cap^g H)\cap K')}(i_{K\cap^g H}^K Z) \in \mathcal{Z}_{i_{K\cap^g H}^{^gH}(^gE)}$$

Since $\mathcal{O}$ admits K/K', $\mathcal{O}$ admits $(K\cap^g H)/((K\cap^g H)\cap K')$ since the admissible sets for $\mathcal{O}$ are closed under restriction in this way. If we knew then that $L_{^gE}$ preserves $i_{^gH}^G\mathcal{O}$-algebras, 4.4.2 would guarantee that $\mathcal{Z}_{i_{K\cap^g H}^{^gH}(^gE)}$ is closed under this norm.

The fact that

$$L_E(-) \text{ preserves } i_H^G\mathcal{O}\text{-algebras} \implies L_{^gE}(-) \text{ preserves } i_{^gH}^G\mathcal{O}\text{-algebras}$$

follows from the fact that the admissible sets for $\mathcal{O}$ are closed under conjugacy, along with the observations

$$N^{K/K'}Z \wedge i_K^H E \simeq * \iff N^{^gK/^g(K')}(^gZ) \wedge i_{^gK}^{^gH}(^gE) \simeq *$$

$$Z \wedge i_K^H E \simeq * \iff {}^gZ \cap i_{^gK}^{^gH}(^gE) \simeq *$$

which follow from the fact that $^g(-) : \mathbf{Sp}^H \to \mathbf{Sp}^{^gH}$ is a symmetric monoidal equivalence of categories. $\square$

Corollary 4.4.4. *If $H \subset G$, and $E \in \mathbf{Sp}^H$ is such that $L_E(-)$ preserves H-commutative rings, then $L_{G_+ \wedge_H E}(-)$ preserves G-commutative rings.*

As in 4.4.1, we will see that the situation is actually better than this: induced localizations automatically upgrade the available norms for an N_∞-algebra, and we make this precise using the results of Section 4.2. We may give $E\mathcal{F}_H$ the trivial Σ_n-action, and as such it becomes the universal $G \times \Sigma_n$-space for the family

$$\mathcal{F}_{H \times \Sigma_n} := \{\Lambda \subset G \times \Sigma_n \, : \, \mathrm{pr}_1(\Lambda) \in \mathcal{F}_H\}$$

where $\mathrm{pr}_1 : G \times \Sigma_n \to G$ is the projection onto the first factor. It is easy to check that if X is any $G \times \Sigma_n$-space, then if we give the G-space $\mathrm{Map}_G(E\mathcal{F}_H, X)$ a $G \times \Sigma_n$-action by postcomposing with the action of Σ_n on X, this is isomorphic as a $G \times \Sigma_n$-space to $\mathrm{Map}_{G \times \Sigma_n}(E\mathcal{F}_H, X)$, where $E\mathcal{F}_H$ has a trivial Σ_n action as above. If $\mathcal{O}$ is any N_∞-operad for the group G, it follows that $\mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O})$ is as well. Moreover, if R is an algebra over $\mathcal{O}$, then $F(E\mathcal{F}_{H+}, R)$ is an algebra over $\mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O})$. 4.2.7 and 4.4.3 together give:

Corollary 4.4.5. *If $R \in \mathbf{Sp}^G$, $E \in \mathbf{Sp}^H$, and $\mathcal{O}$ is an N_∞ operad for the group G such that R is an $\mathcal{O}$-algebra and L_E preserves $i_H^G\mathcal{O}$-algebras, then $L_{G_+ \wedge_H E}(R)$ is a $\mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O})$-algebra.*

In the situation of the corollary, we find that R acquires *more* norms after localizing at $G_+ \wedge_H E$ since the collection of admissible sets for $\mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O})$ contains that of $\mathcal{O}$. We determine now exactly which new norms it acquires. If $\mathcal{O}$ is any N_∞-operad for the group H, then the coinduced operad $F_H(G, \mathcal{O})$ is an N_∞-operad for the group G ([12], 6.14), and we have the following:

Proposition 4.4.6. $\mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O}) \simeq F_H(G, i_H^G\mathcal{O})$ *as N_∞-operads.*

Proof. We have a zig zag of maps of operads

$$\mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O}) \longleftarrow \mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O}) \times F_H(G, i_H^G\mathcal{O}) \longrightarrow F_H(G, i_H^G\mathcal{O})$$

given by the projection maps. It follows that if, for all $n \geq 0$, $\mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O})_n$ and $F_H(G, i_H^G \mathcal{O})_n$ are universal $G \times \Sigma_n$-spaces for the same family of subgroups, then both projections are equivalences.

Let $\mathcal{U}_n$ be the category of universal $(G \times \Sigma_n)$-spaces $E\mathcal{F}$ for $\mathcal{F}$ a family of graph subgroups of $G \times \Sigma_n$. It is an immediate consequence of Elmendorf's theorem that $Ho(\mathcal{U}_n)$ is equivalent to the poset of families of graph subgroups of $G \times \Sigma_n$, via inclusion. Therefore, if $E \in \mathcal{U}_n$, then $E = E\mathcal{F}$ is a universal $G \times \Sigma_n$-space for the family of subgroups

$$\mathcal{F} = \bigcup_{\substack{\mathcal{F}' : \\ \exists E\mathcal{F}' \to E}} \mathcal{F}'$$

given by the union of families $\mathcal{F}'$ having the property that there is a $G \times \Sigma_n$-equivariant map $E\mathcal{F}' \to E$. For $\mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O}_n)$, by adjunction, there is such a map if and only if there is a map

$$E\mathcal{F} \times E\mathcal{F}_H \to \mathcal{O}_n$$

Since $E\mathcal{F} \times E\mathcal{F}_H \simeq E(\mathcal{F} \cap \mathcal{F}_{H \times \Sigma_n})$, this happens if and only if

$$\mathcal{F} \cap \mathcal{F}_{H \times \Sigma_n} \subset \mathcal{F}_{\mathcal{O}_n}$$

One may show that $F_H(G, i_{H \times \Sigma_n}^{G \times \Sigma_n} \mathcal{O}_n) \cong F_{H \times \Sigma_n}(G \times \Sigma_n, i_{H \times \Sigma_n}^{G \times \Sigma_n} \mathcal{O}_n)$ so that there is a $G \times \Sigma_n$-map

$$E\mathcal{F} \to F_H(G, i_{H \times \Sigma_n}^{G \times \Sigma_n} \mathcal{O}_n)$$

if and only if there is a map

$$i_{H \times \Sigma_n}^{G \times \Sigma_n} E\mathcal{F} \to i_{H \times \Sigma_n}^{G \times \Sigma_n} \mathcal{O}_n$$

by adjunction. One checks easily that these are the following universal $H \times \Sigma_n$-spaces

$$i_{H \times \Sigma_n}^{G \times \Sigma_n} E\mathcal{F} = E(\Gamma \subset H \times \Sigma_n : \Gamma \in \mathcal{F})$$

$$i_{H \times \Sigma_n}^{G \times \Sigma_n} \mathcal{O}_n = E(\Gamma \subset H \times \Sigma_n : \Gamma \in \mathcal{O}_n)$$

Hence the map above exists if and only if

$$\{\Gamma \subset H \times \Sigma_n : \Gamma \in \mathcal{F}\} \subset \mathcal{F}_{\mathcal{O}_n}$$

Since $\mathcal{F}_{\mathcal{O}_n}$ is a family and in particular closed under subconjugates, this happens if and only if

$$\mathcal{F} \cap \{\Gamma \subset G \times \Sigma_n : \Gamma \text{ is subconjugate to } H \times \Sigma_n\} \subset \mathcal{F}_{\mathcal{O}_n}$$

It therefore suffices to observe that

$$\mathcal{F}_{H \times \Sigma_n} = \{\Gamma \subset G \times \Sigma_n : \Gamma \text{ is subconjugate to } H \times \Sigma_n\}$$

□

Corollary 4.4.7. *For any $K \subset G$, a K-set T is admissible for* $\mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O})$ *if and only if for all $g \in G$, $i^{gKg^{-1}}_{H \cap gKg^{-1}}{}^{g}T$ is admissible for $\mathcal{O}$. In particular, if $i^G_H \mathcal{O}$ is genuine H-E_∞, then* $\mathrm{Map}_G(E\mathcal{F}_H, \mathcal{O})$ *is genuine G-E_∞.*

Proof. It is clear that if $K \subset H$, and T is a K-set, then $i^G_H \mathcal{O}$ admits T iff $\mathcal{O}$ admits T. Now we apply the previous proposition and [12, 6.16] □

The following is the most direct generalization of 4.4.1 above, in the case where $i^G_H \mathcal{O}$ is genuine H-E_∞.

Corollary 4.4.8. *Let $R \in \mathbf{Sp}^G$ be an algebra over an N_∞-operad $\mathcal{O}$ such that $i^G_H \mathcal{O}$ is a genuine H-E_∞-operad, and let $E \in \mathbf{Sp}^H$. If $L_E(-)$ preserves H-commutative rings, then $L_{G_+ \wedge_H E}(R)$ is a G-commutative ring. In particular, if R is an $\mathcal{O}$-algebra such that $i^G_H \mathcal{O}$ is genuine H-E_∞, then $F(E\mathcal{F}_{H+}, R)$ is a G-commutative ring.*

Proof. Note that $F(E\mathcal{F}_{H+}, R) \simeq L_{G/H_+}(R) \simeq L_{G_+ \wedge_H S^0}(R)$, so that the second assertion follows from the first. For the first assertion, we simply combine 4.4.5 and 4.4.7. □

Example 4.4.9. $L_{E_G(m)}(-)$ sends $\mathcal{O}$-algebras to G-commutative rings for all n and m.

4.5 Restriction of idempotents along a quasi-Galois extension

We digress from categories of G-spectra to highlight the extent to which the results of Section 4.2.3 may be generalized to other settings in which Bousfield localization is possible. This is motivated by the following theorem of Balmer, Dell'Ambroglio, and Sanders:

Theorem 4.5.1. *[5, Theorem 1.1] For $H \subset G$ a subgroup, there is an equivalence of tt-categories*

$$Ho(\mathbf{Sp}^H) \simeq \mathrm{Mod}_{Ho(\mathbf{Sp}^G)}(F(G/H_+, S^0))$$

where the latter is the category of modules in $Ho(\mathbf{Sp}^G)$ over the ring spectrum $F(G/H_+, S^0)$. Under this equivalence, the functor $i_H^G(-)$ corresponds to extension of scalars along the unit map $S^0 \to F(G/H_+, S^0)$, and $G_+ \wedge_H (-) \simeq F_H(G_+, -)$ corresponds to restriction of scalars.

Mathew, Noel, and Naumann upgraded this to an equivalence of symmetric monoidal ∞-categories [66]

$$\mathbf{Sp}^H \simeq \mathrm{Mod}_{\mathbf{Sp}^G}(F(G/H_+, S^0))$$

and studied the extent to which a commutative algebra A in a presentable, symmetric monoidal stable ∞-category $(\mathcal{C}, \otimes, \mathbb{1})$ exhibited categorical properties similar to those seen in equivariant homotopy theory with $A = F(G/H_+, S^0)$. In our context, the analogy suggests that perhaps a smashing A-module M will pull back to an object in $\mathcal{C}$ whose Bousfield localization functor becomes smashing after completion at A. That is, if $\eta : \mathbb{1} \to A$ is the unit map of A, if $M \in \mathrm{Mod}_{\mathcal{C}}(A)$ determines a smashing localization in $\mathrm{Mod}_{\mathcal{C}}(A)$, following 4.2.21, we expect a formula

$$L_{\eta^* M}(-) = L_A(L_{\eta^* M}(\mathbb{1}) \otimes -)$$

in $\mathcal{C}$. However, 4.2.25 tells us that, even in the motivating example $S^0 \to F(G/H_+, S^0)$, we need H to be normal for such a formula to hold. Hence we are led to ask that η be a quasi-Galois extension.

4.5.1 Background on stable ∞-categories and quasi-Galois extensions.

We review what is needed to establish the desired localization formulae for a quasi-Galois extension. We use the language of ∞-categories following [60] and closely follow the discussion in Section 1 of [66], where more detail can be found. In all that follows, we will let

$(\mathcal{C}, \otimes, \mathbb{1})$ be a presentable, symmetric monoidal stable ∞-category in which $- \otimes -$ commutes with colimits in each variable.

Definition 4.5.2. Let $M \in \mathcal{C}$. We let $\mathcal{Z}_M$ be the full subcategory of $\mathcal{C}$ consisting of those $Z \in \mathcal{C}$ such that $Z \otimes M \simeq *$. We let $\mathcal{L}_M$ denote the full subcategory of $\mathcal{C}$ consisting of those $Y \in \mathcal{C}$ such that the space $\mathrm{Map}_{\mathcal{C}}(Z, Y)$ is contractible for all $Z \in \mathcal{Z}_M$.

It follows formally from [60, Section 5.5] that $\mathcal{L}_M$ is a presentable stable ∞-category, and the inclusion $\mathcal{L}_M \hookrightarrow \mathcal{C}$ admits a left adjoint, $L_M(-)$. Moreover, by [59, 2.2.1.9], $\mathcal{L}_M$ inherits the structure of a symmetric monoidal ∞-category so that $L_M : \mathcal{C} \to \mathcal{L}_M$ is symmetric monoidal. The tensor product in $\mathcal{L}_M$ is then necessarily given by the formula

$$L_M(X) \hat{\otimes} L_M(Y) := L_M(X \otimes Y)$$

With this in place, the discussion in Section 2 may be repeated in this setting *mutatis mutandis.* In particular, we may use smashing localizations and tensor idempotents interchangeably (see [59, Section 6.3] or [25, Section 3] for more details).

Suppose now we have an object $A \in \mathrm{CAlg}(\mathcal{C})$ - this induces an Ind-Res adjunction

$$\begin{tikzcd} \mathcal{C} = \mathrm{Mod}_{\mathcal{C}}(\mathbb{1}) \arrow[d, shift right, "\eta_*"'] \\ \mathrm{Mod}_{\mathcal{C}}(A) \arrow[u, shift right, "\eta^*"'] \end{tikzcd}$$

$\mathrm{Mod}_{\mathcal{C}}(A)$ is a presentable, symmetric monoidal stable ∞-category, η_* is a symmetric monoidal functor, and the adjunction $\eta_* \dashv \eta^*$ satisfies the projection formula

$$N \otimes \eta^*(M) \simeq \eta^*(\eta_* N \otimes_A M)$$

(see [66, Section 5.2]). Under the assumption that A is dualizable in $\mathcal{C}$, Mathew, Naumann, and Noel deduce the following description of $\mathcal{L}_A$.

Theorem 4.5.3. *[66, Theorem 2.30] If A is dualizable in $\mathcal{C}$, the functor η_* descends to an equivalence of symmetric monoidal ∞-categories*

$$\mathcal{L}_A \simeq \mathrm{Tot}(\mathrm{Mod}_{\mathcal{C}}(A) \rightrightarrows \mathrm{Mod}_{\mathcal{C}}(A \otimes A) \substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow} \cdots)$$

In our motivating example of $A = F(G/H_+, S^0) \in \mathrm{CAlg}(\mathbf{Sp})$ for $H \triangleleft G$, the double coset formula allows us to identify the simplicial object on the right hand side as the cobar complex computing $(\mathbf{Sp}^H)^{h(G/H)}$. This generalizes to the following situation:

Definition 4.5.4. Let G be a finite group, $R \in \mathrm{CAlg}(\mathcal{C})$, and $A \in \mathrm{Fun}(BG, \mathrm{CAlg}(\mathcal{C})_{R/})$. Consider the diagram in $\mathrm{CAlg}(\mathcal{C})$.

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, bend left, "\Delta^{tw}"] & \\
A \arrow[r] \arrow[drr, bend right, "\Delta"'] & A \otimes_R A \arrow[dr, dashed, "\phi"] & \\
& & \prod\limits_{g \in G} A
\end{tikzcd}$$

where $\pi_g \circ \Delta^{tw} = g : A \to A$. We say that $R \to A$ is a **quasi-Galois extension** if ϕ is an equivalence.

Remark 4.5.5. If we required additionally that the morphism $R \to A^{hG}$ be an equivalence, this would be the usual definition of a Galois extension, due to Rognes [83]. This terminology is used in [75] where quasi-Galois extensions are studied in a tt-geometry context.

As before, we will take $R = \mathbb{1}$, and we record the following immediate consequence of 4.5.3:

Corollary 4.5.6. *If A is dualizable in $\mathcal{C}$, and $\eta : \mathbb{1} \to A$ is a quasi-Galois extension, the functor η_* descends to an equivalence of symmetric monoidal ∞-categories*

$$\mathcal{L}_A \simeq (\mathrm{Mod}_{\mathcal{C}}(A))^{hG}$$

Remark 4.5.7. If η were a Galois extension, the dualizability condition on A would be automatic [83, 6.2.1].

When η is a quasi-Galois extension, the projection formula gives the following decomposition, of which the double-coset formula for $i_H^G(G_+ \wedge_H -)$ is a special case.

Lemma 4.5.8. *For $M \in \mathrm{Mod}_{\mathcal{C}}(A)$,*

$$\eta_* \eta^* M = \bigoplus_{g \in G} {}^g M$$

4.5.2 Smashing A-modules

Our desired localization formulae are of the form

$$L(-) = L_A(L(\mathbb{1}) \otimes -)$$

By definition of the symmetric monoidal structure in $\mathcal{L}_A$, producing a localization functor $L(-)$ on $\mathcal{C}$ given by such a formula is equivalent to producing a smashing localization in $\mathcal{L}_A$. Corollary 4.5.6 tells us that smashing localizations in $\mathcal{L}_A$ are the same thing as smashing localizations in $(\mathrm{Mod}_{\mathcal{C}}(A))^{hG}$. This allows us to produce smashing localizations in $\mathcal{L}_A$ from smashing localizations in $\mathrm{Mod}_{\mathcal{C}}(A)$ via norm functors.

Construction 4.5.9. *Let $(\mathcal{D}, \otimes, \mathbb{1})$ be a presentable, symmetric monoidal ∞-category with G-action (e.g. $\mathrm{Mod}_{\mathcal{C}}(A)$ as above). There is a symmetric monoidal functor*

$$N : \mathcal{D} \to \mathcal{D}^{hG}$$

such that the composite

$$\mathcal{D} \xrightarrow{N} \mathcal{D}^{hG} \hookrightarrow D$$

is given by the functor

$$M \mapsto \bigotimes_{g \in G} {}^g M$$

Remark 4.5.10. There is a right adjoint

$$\mathrm{Fun}(BG, \mathrm{SymMon}\infty\text{-}\mathrm{Cat}) \to G\text{-}\mathrm{SymMon}\infty\text{-}\mathrm{Cat}$$

to the forgetful functor which sends $(\mathcal{D}, \otimes, \mathbb{1})$, a presentable, symmetric monoidal ∞-category with G-action, to the G-symmetric monoidal ∞-category $\underline{\mathcal{D}}(G/H) = \mathcal{D}^{hH}$, with norm map $\underline{\mathcal{D}}(G/e) \to \underline{\mathcal{D}}(G/G)$ as in 4.5.9. This is a higher algebra analog of the functor that sends a commutative ring with G-action to its fixed-point Tambara functor. An account of this construction is to appear in [72].

This construction appears in the context of the symmetric monoidal G-categories of Guillou, May, Merling, and Osorno (see [31, 3.7]), the normed symmetric monoidal categories of Rubin (see [84, 3.7]), and the symmetric monoidal mackey functors of Hill-Hopkins (see [38, 2.6]).

By use of N, we may therefore send an idempotent e in $\mathrm{Mod}_{\mathcal{C}}(A)$ to an idempotent $N(e)$ in $\mathcal{L}_A$. This determines some smashing localization in $\mathcal{L}_A$, and using 6.8, we may identify its corresponding Bousfield class in terms of that of e. We have the following:

Theorem 4.5.11. *Suppose $(\mathcal{C}, \otimes, \mathbb{1})$ and A are as above. In particular, assume $\eta : \mathbb{1} \to A$ is a quasi-Galois extension and A is dualizable in $\mathcal{C}$. If $M \in \mathrm{Mod}_{\mathcal{C}}(A)$ is smashing, then we have the formula in $\mathcal{C}$:*

$$L_{\eta^* M}(-) = L_A(L_{\eta^* M}(\mathbb{1}) \otimes -)$$

Proof. For $X \in \mathcal{C}$, the composite

$$X \to L_{\eta^* M}(\mathbb{1}) \otimes X \to L_A(L_{\eta^* M}(\mathbb{1}) \otimes X)$$

becomes an equivalence after applying $- \otimes \eta^* M$. This is clear for the first map, and for the second map, for any $Y \in \mathcal{C}$, we have a commutative diagram

$$\begin{tikzcd}
Y \otimes \eta^* M \arrow[r] \arrow[d, "\simeq"] & L_A(Y) \otimes \eta^* M \arrow[d, "\simeq"] \\
\eta^*(\eta_* Y \otimes_A M) \arrow[r] & \eta^*(\eta_*(L_A(Y)) \otimes_A M)
\end{tikzcd}$$

by the projection formula. The bottom arrow is an equivalence because $\eta_* Y \to \eta_*(L_A(Y))$ is an equivalence by definition. It suffices now to show that $L_A(L_{\eta^* M}(\mathbb{1}) \otimes X)$ is $\eta^* M$-local in $\mathcal{C}$.

Suppose we knew that $L_A(\eta^*M)$ determined a smashing Bousfield class in $\mathcal{L}_A$. Then if $Z \in \mathcal{Z}_{\eta^*M}$, we have $L_A(Z)\hat{\otimes}L_A(\eta^*M) \simeq *$, and

$$\mathrm{Map}_{\mathcal{C}}(Z, L_A(L_{\eta^*M}(\mathbb{1}) \otimes X)) \simeq \mathrm{Map}_{\mathcal{L}_A}(L_A(Z), L_A(L_{\eta^*M}(\mathbb{1}))\hat{\otimes}L_A(X))$$

Since $L_A(\eta^*M)$-locals form a $\hat{\otimes}$-ideal in $\mathcal{L}_A$ by assumption, it would therefore suffice to show that $L_A(L_{\eta^*M}(\mathbb{1}))$ is $L_A(\eta^*M)$-local in $\mathcal{L}_A$. If $L_A(Z') \in \mathcal{L}_A$ is an $L_A(\eta^*M)$-$\hat{\otimes}$-acyclic, then

$$\mathrm{Map}_{\mathcal{L}_A}(L_A(Z'), L_A(L_{\eta^*M}(\mathbb{1}))) \simeq \mathrm{Map}_{\mathcal{C}}(Z', L_{\eta^*M}(\mathbb{1})) \simeq *$$

The first equivalence is by adjunction and the fact that $\mathcal{L}_{\eta^*M} \subset \mathcal{L}_A$, as $\mathcal{Z}_A \subset \mathcal{Z}_{\eta^*M}$ by the projection formula. For the second, $Z' \otimes \eta^*M$ is A-local, as an A-module, hence

$$Z' \otimes \eta^*M \simeq L_A(Z' \otimes \eta^*M) \simeq L_A(Z')\hat{\otimes}L_A(\eta^*M) \simeq *$$

We now show that $L_A(\eta^*M)$ determines a smashing Bousfield class in $\mathcal{L}_A$. Let $e_M, f_M \in \mathrm{Mod}_{\mathcal{C}}(A)$ denote the left and right idempotents corresponding to M, respectively. If $N(-)$ is the functor in 4.5.9, we have that

$$\ker(-\hat{\otimes}\mathrm{cofib}(N(e_M) \to \mathbb{1}))$$

is a smashing ideal in $\mathcal{L}_A$. It suffices to show that it coincides with $\ker(-\hat{\otimes}L_A(\eta^*M))$. Since

η_* is conservative on $\mathcal{L}_A$, we have

$$\begin{aligned}
Z\hat{\otimes}\mathrm{cofib}(N(e_M)\to \mathbb{1})\simeq * &\iff \eta_*(Z\hat{\otimes}\mathrm{cofib}(N(e_M)\to \mathbb{1}))\simeq *\\
&\iff \eta_*(Z)\otimes_A \mathrm{cofib}(\eta_* N(e_M)\to \mathbb{1})\simeq *\\
&\iff \eta_*(Z)\otimes_A \mathrm{cofib}(\bigotimes_{g\in G} {}^g e_M\to \mathbb{1})\simeq *\\
&\iff \eta_*(Z)\otimes_A \mathrm{cofib}(e_{\bigoplus_{g\in G} {}^g M}\to \mathbb{1})\simeq *\\
&\iff \eta_*(Z)\otimes_A f_{\bigoplus_{g\in G} {}^g M}\simeq *\\
&\iff \eta_*(Z)\otimes_A \bigoplus_{g\in G} {}^g M\simeq *\\
&\iff \eta_*(Z)\otimes_A \eta_*\eta^* M\simeq *\\
&\iff \eta_*(Z\otimes \eta^* M)\simeq *\\
&\iff Z\hat{\otimes}\eta^* M\simeq *
\end{aligned}$$

The third equivalence is by definition of $N(-)$, the fourth and sixth follow as in 4.1.13, and the seventh is 4.5.8. □

Remark 4.5.12. When $\mathcal{C} = \mathbf{Sp}^G$ and $A = F(G/H_+, S^0)$ for $H \lhd G$, this recovers 4.2.21. Moreover, the description of $L_A(-)$ in this case may be generalized: it follows from [66, Proposition 2.21] that in the situation of 4.5.11, we have the formula

$$L_A(Y)\simeq (Y\otimes A)^{hG}\simeq F(\mathbb{D}(A),Y)^{hG}\simeq F(\mathbb{D}(A)_{hG},Y)$$

and so the formula in 4.5.11 may be made more explicit:

$$L_{\eta^*M}(X)\simeq F(\mathbb{D}(A)_{hG}, L_{\eta^*M}(\mathbb{1})\otimes X)$$

Remark 4.5.13. We may derive an analogous formula for a general quasi-Galois extension $\eta: R\to A$ in $\mathcal{C}$ such that A is dualizable in $\mathrm{Mod}_{\mathcal{C}}(R)$: a smashing R-linear A-module $M \in \mathrm{Mod}_{\mathrm{Mod}_{\mathcal{C}}(R)}(A)$ determines a smashing localization in the category of A-locals in $\mathrm{Mod}_{\mathcal{C}}(R)$ corresponding to the Bousfield class of $\eta^* M$. The same proofs work since R is the unit in $\mathrm{Mod}_{\mathcal{C}}(R)$.

We have also in this setting a necessary and sufficient condition for η^*M to be smashing in $\mathcal{C}$, i.e. a generalization of 4.2.20. In [66], the role of the geometric fixed points functor is generalized to this setting as follows: consider the cofiber sequence

$$\mathbb{D}(A) \xrightarrow{\mathbb{D}(\eta)} \mathbb{1} \xrightarrow{a_A} C(A)$$

Define

$$U_A \coloneqq \operatorname{colim}(\mathbb{1} \xrightarrow{a_A} C(A) \xrightarrow{a_A} C(A)^{\otimes 2} \xrightarrow{a_A} \cdots)$$

Then U_A is a right idempotent in $\mathcal{C}$, and for any $X \in \mathcal{C}$, there is a homotopy pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \otimes U_A \arrow[d] \\ L_A(X) \arrow[r] & L_A(X) \otimes U_A \end{tikzcd}$$

Proposition 4.5.14. *In the situation of 4.5.11, η^*M is smashing in $\mathcal{C}$ if and only if $L_{\eta^*M}(\mathbb{1}) \otimes U_A \simeq *$.*

Proof. If η^*M is smashing, then $L_{\eta^*M}(\mathbb{1}) \otimes U_A \simeq L_{\eta^*M}(U_A)$, but $\mathcal{Z}_A \subset \mathcal{Z}_{\eta^*M}$, and in the cofiber sequence

$$\mathbb{D}(A) \otimes A \to A \to C(A) \otimes A$$

the first map splits via the map $A \to \mathbb{D}(A) \otimes A$ adjoint to the multiplication map $A \otimes A \to A$. Therefore the map $A \to C(A) \otimes A$ is null, and so

$$U_A \otimes A \simeq \operatorname{colim}(A \to C(A) \otimes A \to C(A)^{\otimes 2} \otimes A \to \cdots) \simeq *$$

Conversely, suppose $L_{\eta^*M}(\mathbb{1}) \otimes U_A \simeq *$, we will show that $L_{\eta^*M}(\mathbb{1}) \otimes X \in \mathcal{L}_{\eta^*M}$ for all $X \in \mathcal{C}$. As above, we have a pullback square

$$\begin{tikzcd} L_{\eta^*M}(\mathbb{1}) \otimes X \arrow[r] \arrow[d] & L_{\eta^*M}(\mathbb{1}) \otimes X \otimes U_A \arrow[d] \\ L_A(L_{\eta^*M}(\mathbb{1}) \otimes X) \arrow[r] & L_A(L_{\eta^*M}(\mathbb{1}) \otimes X) \otimes U_A \end{tikzcd}$$

By 4.5.11, we may identify the bottom row with the map $L_{\eta^*M}(X) \to L_{\eta^*M}(X) \otimes U_A$. By assumption, $L_{\eta^*M}(\mathbb{1}) \otimes U_A \simeq *$, so $L_{\eta^*M}(X) \otimes U_A$ is contractible as a module over $L_{\eta^*M}(\mathbb{1}) \otimes U_A$. Therefore the left hand arrow is an equivalence and the target is η^*M-local. $\square$

Corollary 4.5.15. *In the situation of 4.5.11, $\eta^* M$ is smashing in $\mathcal{C}$ if and only if $L_{\eta^* M}(\mathbb{1})$ is in the thick subcategory of $\mathcal{C}$ generated by A.*

Proof. This follows immediately from [66, Theorem 4.19]. □

Example 4.5.16. Let $H \triangleleft G$ be a closed normal subgroup of finite index in a compact Lie group G. If $E \in \mathbf{Sp}^H$ is smashing, then

$$L_{G_+ \wedge_H E}(X) = F(E\mathcal{F}_{H+}, L_E(S^0) \wedge X)$$

for all $X \in \mathbf{Sp}^G$.

Proof. In [66, pg. 29], it is noted that the the analog of 4.3.1 (and its ∞-categorical refinement) hold in this setting, and the formula now follows from 4.5.12. □

Example 4.5.17. In fact, our above arguments may be used to show the analogous version of 4.2.21 for induced localizations holds for any of the equivariant tt-categories studied in [5].

Chapter 5

COFREENESS IN REAL-ORIENTED HOMOTOPY THEORY

In the previous chapter, we investigated analogues of the Ravenel conjectures in C_{2^n}-equivariant homotopy theory, where Real-oriented homotopy theory played the role of chromatic homotopy theory. Our main findings showed that many of the same theorems hold, but one has to pass to cofree G-spectra. In this chapter, we clarify the role of cofreeness in the context of Real-oriented homotopy theory by generalizing a result of Hu-Kriz. Hu and Kriz showed in [48] that Real bordism theory - $MU_{\mathbb{R}}$ - is cofree, i.e. the map

$$MU_{\mathbb{R}} \to F(EC_{2+}, MU_{\mathbb{R}})$$

is an equivalence of C_2-spectra. We generalize this to the norms of $MU_{\mathbb{R}}$, proving that, for all $n \geq 1$, $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ is cofree, i.e. the map

$$N_{C_2}^{C_{2^n}} MU_{\mathbb{R}} \to F(EC_{2^n+}, N_{C_2}^{C_{2^n}} MU_{\mathbb{R}})$$

is an equivalence of C_{2^n}-spectra. The equivariant spectra $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ play a central role in the solution to the Kervaire Invariant One problem by Hill, Hopkins, and Ravenel [39]. Their detecting spectrum Ω is the homotopy fixed point spectrum of a localization $\Omega_{\mathbb{O}} := D^{-1} N_{C_2}^{C_8} MU_{\mathbb{R}}$ of $N_{C_2}^{C_8} MU_{\mathbb{R}}$. An essential piece of their argument is their homotopy fixed point theorem [39, 1.10], which states that this homotopy fixed point spectrum coincides with the *genuine* fixed point spectrum, i.e. that $\Omega_{\mathbb{O}}$ is cofree. Our result shows that this holds even *before* localization away from D.

We use the above cofreeness result to give a new, more conceptual proof of a fundamental and deep result in equivariant homotopy theory, the Segal Conjecture for C_2:

Theorem 5.0.1. *For any bounded below spectrum X, the Tate diagonal*

$$X \to (N_e^{C_2}(X))^{tC_2}$$

is a 2-complete equivalence.

See Section 3.4 for more details on the relationship between the Tate diagonal and the Segal conjecture for C_2. Lin's proof involves a difficult calculation of a continuous Ext group

$$\widehat{\mathrm{Ext}}_{\mathcal{A}}(H^*(\mathbb{RP}^\infty_{-\infty};\mathbb{F}_2);\mathbb{F}_2)$$

where $\mathcal{A}$ is the Steenrod algebra. Nikolaus and Scholze showed, however, that 5.0.1 follows formally for all X bounded below from the case $X = H\mathbb{F}_2$. Hahn and Wilson [35] used this to show that 5.0.1 can be established by analysis of the descent spectral sequence for the map

$$N_e^{C_2} H\mathbb{F}_2 \to H\underline{\mathbb{F}_2}$$

which reduces to a continuous Ext group calculation over a much smaller polynomial coalgebra $\mathbb{F}_2[x]$.

We give a proof of Lin's Theorem that involves essentially no homological algebra and proceeds from a chromatic approach. Essential to our proof is the identification $\Phi^{C_2}(N_{C_2}^{C_4} BP_{\mathbb{R}}) \simeq N_e^{C_2} H\mathbb{F}_2$. In [67], Meier, Shi, and Zeng use this identification to deduce differentials in the homotopy fixed point spectral sequence of $N_e^{C_2} H\mathbb{F}_2$ from differentials in the slice spectral sequence of $N_{C_2}^{C_4} BP_{\mathbb{R}}$, thus establishing a connection between the Segal Conjecture and the HHR slice theorem. We make this connection precise by proving the following:

Theorem 5.0.2. *For any $n > 1$, the cofreeness of $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ is equivalent to Lin's Theorem together with the cofreeness of $MU_{\mathbb{R}}$.*

Theorem 5.0.2 is a formal consequence of the Nikolaus-Scholze Tate orbit lemma ([74], I.2.1), and this gives a straightforward proof of the cofreeness of $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ using Lin's Theorem and the result of Hu and Kriz. On the other hand, we give an independent proof of the cofreeness of $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ that works for *all* $n > 0$, following a chromatic approach which depends only on the HHR slice theorem (see Section 3.3).

We give a sketch here of this proof in the case $n = 1$. The idea is that $BP_{\mathbb{R}}[\overline{v_i}^{-1}]$ is cofree for formal reasons, so one can take an approach via local cohomology and form cartesian cubes

$$\begin{tikzcd}
\tilde{L}_2 BP_{\mathbb{R}} \arrow[r] \arrow[d] & BP_{\mathbb{R}}[\overline{v_1}^{-1}] \arrow[d] \\
BP_{\mathbb{R}}[\overline{v_2}^{-1}] \arrow[r] & BP_{\mathbb{R}}[(\overline{v_1 v_2})^{-1}]
\end{tikzcd}$$

$$\begin{tikzcd}
 & BP_{\mathbb{R}}[\overline{v_3}^{-1}] \arrow[rr] \arrow[dd] & & BP_{\mathbb{R}}[(\overline{v_2 v_3})^{-1}] \arrow[dd] \\
\tilde{L}_3 BP_{\mathbb{R}} \arrow[ur] \arrow[rr] \arrow[dd] & & BP_{\mathbb{R}}[\overline{v_2}^{-1}] \arrow[ur] \arrow[dd] & \\
 & BP_{\mathbb{R}}[(\overline{v_1 v_3})^{-1}] \arrow[rr] & & BP_{\mathbb{R}}[(\overline{v_1 v_2 v_3})^{-1}] \\
BP_{\mathbb{R}}[\overline{v_1}^{-1}] \arrow[ur] \arrow[rr] & & BP_{\mathbb{R}}[(\overline{v_1 v_2})^{-1}] \arrow[ur] &
\end{tikzcd}$$

and so on, and $\tilde{L}_n BP_{\mathbb{R}}$ is cofree for all n. Applying the slice tower to each vertex $BP_{\mathbb{R}}[(\overline{v_{i_1}} \cdots \overline{v_{i_j}})^{-1}]$, one forms a cartesian cube in filtered C_2-spectra, and the limit term gives a modified slice filtration of $\tilde{L}_n BP_{\mathbb{R}}$. It is then a formal consequence of the HHR slice theorem that, taking the limit in n, one recovers the slice tower of $BP_{\mathbb{R}}$.

Remark 5.0.3. Our results should shed light on the spectral sequences studied in Meier, Shi, and Zeng [67]. In particular, the map from the slice spectral sequence of $N_{C_2}^{C_4} BP_{\mathbb{R}}$ to its HFPSS (see Remark 3.3.4) is an isomorphism below a line of slope 1 (see [89]). The slice spectral sequence vanishes above a line of slope 3, but there are many classes above this line in the HFPSS. Since $N_{C_2}^{C_4} BP_{\mathbb{R}}$ is cofree, the map between them must give an isomorphism on their E_∞-pages, so there must be some pattern of differentials killing all the classes above this line in the HFPSS.

In Section 5.1, we show that the cofreeness of $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ follows formally from (and is equivalent to) the Hu-Kriz $n = 1$ case together with Lin's Theorem. This is the most direct way to our cofreeness result, using these known results. In Section 5.2, we withhold knowledge of these theorems and give a different proof - via chromatic hypercubes - that $N_{C_2}^{C_4} BP_{\mathbb{R}}$ is cofree. In turn, this result implies the $n = 1$ case and Lin's theorem, which then gives the result for $n > 2$ by the same induction used in Section 5.1. We use the notation $MU^{((G))}$ and $BP^{((G))}$ to denote $N_{C_2}^{G} MU_{\mathbb{R}}$ and $N_{C_2}^{G} BP_{\mathbb{R}}$ respectively, as in HHR.

5.1 Cofreeness and gluing maps

5.1.1 Cofreeness

We begin by reviewing the notion of cofreeness for a genuine G-spectrum (see Remark 3.1.11).

Proposition 5.1.1. *For $X \in \mathbf{Sp}^G$, the following are equivalent*

1. *$X \to F(EG_+, X)$ is an equivalence of G-spectra, i.e. X is a cofree G-spectrum as in Remark 3.1.11.*
2. *$X^H \to X^{hH}$ is an equivalence of spectra for all $H \subset G$.*
3. *X is G_+-local.*

Proof. For $1 \iff 3$, it suffices to show that $L_{G_+}(X) = F(EG_+, X)$. The map

$$X \to F(EG_+, X)$$

becomes an equivalence after smashing with G_+ by the Frobenius relation, and the target is G_+-local because if $Z \wedge G_+ \simeq *$, then

$$[Z, F(EG_+, X)]^G = [Z \wedge EG_+, X]^G = 0$$

as EG_+ is in the localizing subcategory generated by G_+. $1 \iff 2$ follows from the fact that the fixed point functors $(-)^H$ are jointly conservative, and

$$i_H^G(F(EG_+, X)) = F(EH_+, i_H^G X)$$

as can be seen from the more general statement

$$i_H^G(L_E(X)) = L_{i_H^G E}(i_H^G X)$$

(see Proposition 4.2.2). □

Definition 5.1.2. We say a G-spectrum X is *cofree* if any of the equivalent conditions in 5.1.1 hold.

Corollary 5.1.3. *The category of cofree G-spectra is closed under homotopy limits.*

Proof. This is true of any category of E-locals. □

We will make use of the slice filtration on G-spectra, as in Section 3.3. Let $X \geq n$ denote that a G-spectrum is slice $\geq n$, i.e. X is slice $(n-1)$-connected. We need the following useful lemma:

Lemma 5.1.4. *Suppose $\{X_i\}_{i\in\mathbb{N}}$ is a family of G-spectra such that, for all $n \in \mathbb{Z}$, all but finitely many X_i have the property that $X_i \geq n$. Then the canonical map*

$$\bigvee_i X_i \to \prod_i X_i$$

is an equivalence.

Proof. It suffices to show that, for all $k \in \mathbb{Z}$, the map of Mackey functors

$$\bigoplus_i \underline{\pi}_k(X_i) \cong \underline{\pi}_k\left(\bigvee_i X_i\right) \to \underline{\pi}_k\left(\prod_i X_i\right) \cong \prod_i \underline{\pi}_k(X_i)$$

is an isomorphism. This follows immediately from the observation that $\underline{\pi}_k(X_i) = 0$ for all but finitely many i. Indeed, by [39, 4.40], if $Y \geq n$, then $\underline{\pi}_k(Y) = 0$ for $k < \lfloor n/|G| \rfloor$ when $n \geq 0$ and for $k < n$ when $n \leq 0$. □

Proposition 5.1.5. *If $MU_{\mathbb{R}}$ is cofree, then $MU_{\mathbb{R}}^{\wedge n}$ is cofree for all $n \geq 1$, and similarly for $BP_{\mathbb{R}}^{\wedge n}$.*

Proof. We proceed by induction on n. Since $MU_{\mathbb{R}}^{\wedge(n-1)}$ is Real-oriented, we have

$$MU_{\mathbb{R}}^{\wedge n} = MU_{\mathbb{R}}^{\wedge(n-1)}[\overline{b_1}, \overline{b_2}, \ldots] = \bigvee_{m \in M} S^{\frac{|m|}{2}\rho} \wedge MU_{\mathbb{R}}^{\wedge(n-1)}$$

where M is a monomial basis of $\mathbb{Z}[b_1, b_2, \ldots]$, applying Proposition 3.2.8. By the lemma, the canonical map

$$\bigvee_{m \in M} S^{\frac{|m|}{2}\rho} \wedge MU_{\mathbb{R}}^{\wedge(n-1)} \to \prod_{m \in M} S^{\frac{|m|}{2}\rho} \wedge MU_{\mathbb{R}}^{\wedge(n-1)}$$

is an equivalence, as $MU_{\mathbb{R}}^{\wedge(n-1)} \geq 0$ and $S^{k\rho} \geq 2k$, so that $S^{k\rho} \wedge MU_{\mathbb{R}}^{\wedge(n-1)} \geq 2k$ by [39, 4.2.6].

This completes the proof, as the category of cofree C_2-spectra is closed under limits and smashing with a dualizable C_2-spectrum, hence the target is cofree. □

5.1.2 Gluing maps

We set up an inductive argument to prove that $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ is cofree. To fix notation, we use $\Phi^{C_{p^k}}$ to denote the functor $\mathbf{Sp}^{C_{p^n}} \to \mathbf{Sp}$ and $\widetilde{\Phi}^{C_{p^k}}$ to denote the functor $\mathbf{Sp}^{C_{p^n}} \to \mathbf{Sp}^{C_{p^{n-k}}}$, so that $i_e^{C_{p^{n-k}}} \circ \widetilde{\Phi}^{C_{p^k}} = \Phi^{C_{p^k}}$. Nikolaus and Scholze use a result of Hesselholt and Madsen [36, 2.1] along with their Tate orbit lemma, to show the following.

Proposition 5.1.6. *[74, Corollary II.4.7] If $X \in \mathbf{Sp}^{C_{p^n}}$ has the property that $\Phi^{C_{p^k}} X \in \mathbf{Sp}$ is bounded below for all $0 \leq k < n$, there is a homotopy limit diagram*

$$\begin{tikzcd}
X^{C_{p^n}} \arrow[rrrr] \arrow[dddd] & & & & \Phi^{C_{p^n}} X \arrow[d] \\
& & & (\widetilde{\Phi}^{C_{p^{n-1}}} X)^{hC_p} \arrow[r] \arrow[d] & (\widetilde{\Phi}^{C_{p^{n-1}}} X)^{tC_p} \\
& & (\widetilde{\Phi}^{C_{p^2}} X)^{hC_{p^{n-2}}} \arrow[r] \arrow[d] & \cdots & \\
& \left(\widetilde{\Phi}^{C_p} X\right)^{hC_{p^{n-1}}} \arrow[r] \arrow[d] & \left((\widetilde{\Phi}^{C_p} X)^{tC_p}\right)^{hC_{p^{n-2}}} & & \\
X^{hC_{p^n}} \arrow[r] & \left(X^{tC_p}\right)^{hC_{p^{n-1}}} & & &
\end{tikzcd}$$

Theorem 5.1.7. *Let Y be a bounded below C_p-spectrum. If $Y^{\wedge p^k}$ is a cofree C_p-spectrum for all $0 \leq k < n$, then $N_{C_p}^{C_{p^n}} Y$ is cofree.*

Proof. Set $X := N_{C_p}^{C_{p^n}} Y$. We proceed by induction on n, with the base case $n = 1$ being tautological. For all $1 \leq k < n$,

$$i_{C_{p^{n-k}}}^{C_{p^n}} X = N_{C_p}^{C_{p^{n-k}}} (Y^{\wedge p^k})$$

is cofree by induction, so it suffices to show the map $X^{C_{p^n}} \to X^{hC_{p^n}}$ is an equivalence. Since Y is bounded below, so is X, and this map is an equivalence if all of the short vertical maps in 5.1.6 are equivalences. Each such map is of the form

$$(f)^{hC_{p^{n-k}}} : \left(\widetilde{\Phi}^{C_{p^k}} X\right)^{hC_{p^{n-k}}} \to \left((\widetilde{\Phi}^{C_{p^{k-1}}} X)^{tC_p}\right)^{hC_{p^{n-k}}}$$

for $k > 0$, which is induced by the map in $\mathbf{Sp}^{C_{p^{n-k}}}$

$$f : \widetilde{\Phi}^{C_{p^k}} X \to (\widetilde{\Phi}^{C_{p^{k-1}}} X)^{tC_p}$$

It therefore suffices to show that f is an equivalence of Borel $C_{p^{n-k}}$-spectra for all $k > 0$, which by definition is simply an underlying equivalence. The underlying map is the natural map

$$\Phi^{C_p}\left(i_{C_p}^{C_{p^{n-k+1}}} \tilde{\Phi}^{C_{p^{k-1}}} X\right) \to \left(i_{C_p}^{C_{p^{n-k+1}}} \tilde{\Phi}^{C_{p^{k-1}}} X\right)^{tC_p}$$

so it suffices to show $i_{C_p}^{C_{p^{n-k+1}}} \tilde{\Phi}^{C_{p^{k-1}}} X$ is a cofree C_p-spectrum. When $k = 1$, we have

$$i_{C_p}^{C_{p^{n-k+1}}} \tilde{\Phi}^{C_{p^{k-1}}} X \simeq Y^{\wedge p^{n-1}}$$

and for $k > 1$, one has

$$i_{C_p}^{C_{p^{n-k+1}}} \tilde{\Phi}^{C_{p^{k-1}}} X \simeq i_{C_p}^{C_{p^{n-k+1}}} (N_e^{C_{p^{n-k+1}}} (\Phi^{C_p} Y)) \simeq N_e^{C_p}\left(\Phi^{C_p}(Y^{\wedge p^{n-k}})\right)$$

using the identification $\widetilde{\Phi}^{C_{p^k}} X \simeq N_e^{C_{p^{n-k}}} (\Phi^{C_p} Y)$ (see [67, Theorem 2.2]). The C_p-spectrum $N_e^{C_p} \Phi^{C_p}(Y^{\wedge p^{n-k}})$ is cofree by the Segal Conjecture for C_p: since $Y^{\wedge p^{n-k}}$ is bounded below and cofree,

$$\Phi^{C_p}(Y^{\wedge p^{n-k}}) \simeq (Y^{\wedge p^{n-k}})^{tC_p}$$

is bounded below and p-complete. □

Remark 5.1.8. This result has various converses. For example, if Y is a bounded below C_p-spectrum, then $N_{C_p}^{C_{p^k}} Y$ is cofree for all $1 \le k \le n$ if and only if $Y^{\wedge p^k}$ is a cofree C_p-spectrum for all $0 \le k < n$. The other direction follows because if $N_{C_p}^{C_{p^{k+1}}} Y$ is cofree, then $Y^{\wedge p^k} = i_{C_p}^{C_{p^{k+1}}} N_{C_p}^{C_{p^{k+1}}} Y$ is also cofree.

If Y is also a ring spectrum, then the direct converse of 5.1.7 is true: $N_{C_p}^{C_{p^n}} Y$ is cofree if and only if $Y^{\wedge p^k}$ is a cofree C_p-spectrum for all $0 \le k < n$. This follows because $Y^{\wedge p^k}$ is a retract of $Y^{\wedge p^{n-1}} = i_{C_p}^{C_{p^n}} N_{C_p}^{C_{p^n}} Y$ in this case.

Corollary 5.1.9. *For all $n \ge 1$, $MU^{((C_{2^n}))}$ is cofree, and similarly for $BP^{((C_{2^n}))}$.*

Proof. $MU_{\mathbb{R}}$ is bounded below, so this follows immediately from 5.1.5, the Hu-Kriz $n = 1$ case, and the theorem. ☐

We have shown that the case $n = 1$, due to Hu and Kriz, along with Lin's theorem, implies that $MU^{((C_{2^n}))}$ is cofree for all $n \ge 1$. The argument can be reversed to point to another proof of Lin's theorem, namely:

Proposition 5.1.10. *For any $n > 1$, the cofreeness of $MU^{((C_{2^n}))}$ implies both Lin's theorem and the $n = 1$ case.*

Proof. If for any $n > 1$, $MU^{((C_{2^n}))}$ is cofree, then a smash power of $BP^{((C_4))}$ is cofree, and it follows that $BP^{((C_4))}$ is cofree, as a retract; similarly for $BP_{\mathbb{R}}$ and therefore for its smash powers by 5.1.5. In this case, the limit diagram in 5.1.6 is as follows:

$$\begin{tikzcd}
(BP^{((C_4))})^{C_4} \arrow[rr] \arrow[dd] & & \Phi^{C_4}(BP^{((C_4))}) \arrow[d] \\
& (\widetilde{\Phi}^{C_2} BP^{((C_4))})^{hC_2} \arrow[r] \arrow[d] & (\widetilde{\Phi}^{C_2} BP^{((C_4))})^{tC_2} \\
(BP^{((C_4))})^{hC_4} \arrow[r] & ((BP^{((C_4))})^{tC_2})^{hC_2} &
\end{tikzcd}$$

The lefthand vertical arrow is an equivalence by assumption, and the middle arrow is an equivalence since $BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$ is cofree. We find that the righthand vertical map is an equivalence, and this is the Tate diagonal $H\mathbb{F}_2 \to (N_e^{C_2} H\mathbb{F}_2)^{tC_2}$, which is an equivalence if and only if Lin's theorem holds, by [74, III.1.7]. $\square$

5.2 Localizations of norms of Real bordism theory

In this section, we give a proof that $N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}$ is cofree that is independent of both Lin's theorem and the Hu-Kriz $n = 1$ case. Our strategy is to show that $BP^{((C_4))}$ is cofree by mimicking the argument sketched in the introduction to show that $BP_{\mathbb{R}}$ is cofree. By 5.1.10, this implies Lin's theorem as well as the cofreeness of $MU_{\mathbb{R}}$, which gives the cases $n > 2$ by 5.1.7.

To construct hypercubes analogous to those for $BP_{\mathbb{R}}$, we need a family of elements in $\pi_{\bigstar}^{C_4} BP^{((C_4))}$ to play the role of the $\overline{v_i}$'s, and we need $BP^{((C_4))}$ to become cofree upon inverting these elements. Following the discussion in [39, Section 6], in $\pi_*^u(BP^{((C_4))}) = \pi_*(BP \wedge BP)$, there are classes $\{t_i\}_{i \geq 1}$ with the property that

$$\pi_*^u(BP^{((C_4))}) = \mathbb{Z}_{(2)}[t_i, \gamma(t_i) \,:\, i \geq 1]$$

as a C_4-algebra, where γ is the generator of C_4 and $\gamma^2(t_i) = -t_i$. The restriction map

$$\pi_{*\rho_2}^{C_2}(BP^{((C_4))}) \to \pi_{2*}^u(BP^{((C_4))})$$

is an isomorphism. Lifting the classes t_i along this map, we have classes

$$\overline{t_i} \in \pi_{(2^i-1)\rho_2}^{C_2}(BP^{((C_4))})$$

and using the C_4-commutative ring structure on $MU_{(2)}^{((C_4))}$, this gives classes

$$N_{C_2}^{C_4}(\overline{t_i}) \in \pi_{(2^i-1)\rho_4}^{C_4}(BP^{((C_4))})$$

Inverting these classes, we may form hypercubes whose limits

$$\tilde{L}_n BP^{((C_4))} := \operatorname{holim}_{\{i_1,\ldots,i_j\} \in \mathcal{P}_0([n])}(BP^{((C_4))}[N_{C_2}^{C_4}(\overline{t_{i_1}} \cdots \overline{t_{i_j}})^{-1}])$$

are easily shown to be cofree. It suffices then to establish that the natural map

$$BP^{((C_4))} \to \mathrm{holim}_n(\tilde{L}_n BP^{((C_4))})$$

is an equivalence. We determine the slice towers of each of the vertices

$$BP^{((C_4))}_{i_1,\ldots,i_j} := BP^{((C_4))}[N_{C_2}^{C_4}(\overline{t_{i_1}}\cdots\overline{t_{i_j}})^{-1}]$$

which determines a filtration on $\tilde{L}_n BP^{((C_4))}$. We therefore analyze the above map in the category of *filtered* C_4-spectra. We show that the associated graded of the filtration on $\tilde{L}_n BP^{((C_4))}$ splits as

$$(H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}) \oplus \mathcal{X}_n$$

where $(H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W})$ is the associated graded of the slice filtration on $BP^{((C_4))}$, and the map $\mathcal{X}_n \to \mathcal{X}_{n-1}$ is null, from which the result follows.

In Section 5.2.1, we begin with some general results on hypercubes that will allow us to deduce the associated graded of the filtration on $\tilde{L}_n BP^{((C_4))}$. In Section 5.2.2, we show that the functor sending a G-spectrum to its slice tower commutes with filtered colimits, allowing us to easily deduce the slice tower of $BP^{((C_4))}_{i_1,\ldots,i_j}$ from that of $BP^{((C_4))}$. We finish in Section 5.2.3 by showing the results of Section 5.2.1 apply - on associated graded - to the hypercubes discussed above, completing the proof.

5.2.1 Generalities on hypercubes

We give some general results on hypercubes that look like (summands of) our chromatic hypercubes for $BP^{((C_4))}$, on associated graded. In this section, we use the language of ∞-categories following [60]; in particular, we work in the model of quasicategories, and use stable ∞-categories following [59]. For a discussion of cubical diagrams in the context of ∞-categories, see [59, Chapter 6], or [1].

We fix $\mathcal{C}$ a stable ∞-category. Let $[n]$ denote the totally ordered set $\{1,\ldots,n\}$, and for T a totally ordered set, let $\mathcal{P}(T)$ denote its power set regarded as a poset under inclusion. Let $\mathcal{P}_0(T)$ denote the sub-poset $\mathcal{P}(T) \setminus \{\varnothing\}$.

Definition 5.2.1. An *n-cube* $\mathcal{X}$ in $\mathcal{C}$ is a functor $\mathcal{X} : \mathcal{P}([n]) \to \mathcal{C}$, and a *partial n-cube* is a functor $\mathcal{P}_0([n]) \to \mathcal{C}$. We say an n-cube $\mathcal{X}$ is *cartesian* if the map

$$\mathcal{X}(\varnothing) \to \operatorname{holim}_{T \in \mathcal{P}_0([n])} \mathcal{X}(T)$$

is an equivalence.

Construction 5.2.2. *Let $\mathcal{P}$ be a poset and objects $C_T \in \mathcal{C}$ for $T \in \mathcal{P}$ given. Regarding $C : T \mapsto C_T$ as a functor from the discrete category* $\mathrm{ob}\mathcal{P}$*, we obtain a diagram $\mathcal{X}_C : \mathcal{P} \to \mathcal{C}$ via left Kan extension along the inclusion* $\mathrm{ob}\mathcal{P} \to \mathcal{P}$*. Concretely,*

$$\mathcal{X}_C(T) = \bigoplus_{S \leq T} C_S$$

and the maps in $\mathcal{X}_C$ are the canonical inclusions.

Definition 5.2.3. When $\mathcal{P} = \mathcal{P}_0([n])$, we say a partial n-cube $\mathcal{X} : \mathcal{P}_0([n]) \to \mathcal{C}$ is *built from disjoint split inclusions* if $\mathcal{X}$ is equivalent to some $\mathcal{X}_C$ as in 5.2.2. If $\mathcal{X}$ is a cartesian n-cube such that the corresponding partial n-cube is built from disjoint split inclusions, we say $\mathcal{X}$ is a *cartesian n-cube built from disjoint split inclusions*.

To make this definition clearer, note that any partial 2-cube built from disjoint split inclusions is equivalent to one of the form

$$\begin{tikzcd}
 & C_2 \arrow[d] \\
C_1 \arrow[r] & C_1 \oplus C_2 \oplus C_{12}
\end{tikzcd}$$

and any partial 3-cube built from disjoint split inclusions is equivalent to one of the form

$$\begin{tikzcd}
 & C_3 \arrow[rr] \arrow[dd] & & C_2 \oplus C_3 \oplus C_{23} \arrow[dd] \\
 & & C_2 \arrow[ur] \arrow[dd] & \\
 & C_1 \oplus C_3 \oplus C_{13} \arrow[rr, crossing over] & & C_1 \oplus C_2 \oplus C_3 \oplus C_{12} \oplus C_{13} \oplus C_{23} \oplus C_{123} \\
C_1 \arrow[ur] \arrow[rr] & & C_1 \oplus C_2 \oplus C_{12} \arrow[ur] &
\end{tikzcd}$$

where the inclusions are the canonical ones. We want to identify the limit of a diagram of this form, and we use a result of Antolin-Camarena and Barthel on computing limits of cubical diagrams inductively:

Proposition 5.2.4. *[1, 2.4] Let $\mathcal{X} : \mathcal{P}_0([n]) \to \mathcal{C}$ be a partial n-cube in $\mathcal{C}$. One has a pullback square*

$$\begin{tikzcd}
\mathrm{holim}_{S\in\mathcal{P}_0([n])}\mathcal{X}(S) \arrow[r] \arrow[d] & \mathrm{holim}_{S\in\mathcal{P}_0([n-1])}\mathcal{X}(S) \arrow[d] \\
\mathcal{X}(\{n\}) \arrow[r] & \mathrm{holim}_{S\in\mathcal{P}_0([n-1])}\mathcal{X}(S\cup\{n\})
\end{tikzcd}$$

Proposition 5.2.5. *Let $\mathcal{X}$ be a partial n-cube in $\mathcal{C}$ built from disjoint split inclusions with respect to some choice of objects $\{C_T\}_{T\in\mathcal{P}_0([n])}$ as in 5.2.2. Then $\mathcal{X}$ satisfies*

1. $\mathrm{holim}_{S\in\mathcal{P}_0([n])}\mathcal{X}(S) \simeq \Omega^{n-1}C_{\{1,\dots,n\}}$

2. *The map*

$$\mathrm{holim}_{S\in\mathcal{P}_0([n])}\mathcal{X}(S) \to \mathrm{holim}_{S\in\mathcal{P}_0([n-1])}\mathcal{X}(S)$$

   is nullhomotopic.

Proof. We proceed by induction on n. For $n = 1$, a cartesian 1-cube is an equivalence

$$\mathrm{holim}_{S\in\mathcal{P}_0([1])}\mathcal{X}(S) \xrightarrow{\simeq} \mathcal{X}(\{1\})$$

and the map in (2) is the map to the terminal object. It is straightforward to show that the partial $(n-1)$-cube

$$\mathcal{P}_0([n-1]) \to \mathcal{P}_0([n]) \xrightarrow{\mathcal{X}} \mathcal{C}$$

is built from disjoint split inclusions, and

$$\mathcal{P}_0([n-1]) \xrightarrow{-\cup\{n\}} \mathcal{P}_0([n]) \xrightarrow{\mathcal{X}} \mathcal{C}$$

is of the form $C_{\{n\}} \oplus \mathcal{Z}$ where $\mathcal{Z}$ is a partial $(n-1)$-cube built from disjoint split inclusions using the objects $\{C_T \oplus C_{T\cup\{n\}}\}_{T\in\mathcal{P}_0([n-1])}$, as in 5.2.2. By induction, 5.2.4 gives a pullback square

$$\begin{tikzcd}
\operatorname{holim}_{S\in\mathcal{P}_0([n])}\mathcal{X}(S) \arrow[r] \arrow[d] & \Omega^{n-2}C_{\{1,\dots,n-1\}} \arrow[d] \\
C_{\{n\}} \arrow[r] & C_{\{n\}} \oplus \Omega^{n-2}C_{\{1,\dots,n-1\}} \oplus \Omega^{n-2}C_{\{1,\dots,n\}}
\end{tikzcd}$$

which is a cartesian 2-cube built from disjoint split inclusions. It therefore suffices to prove the proposition in the case $n = 2$, which is the claim that for objects $C_1, C_2, C_{12} \in \mathcal{C}$, there is a pullback square of the form

$$\begin{tikzcd}
\Omega C_{12} \arrow[r, "0"] \arrow[d, "0"'] & C_2 \arrow[d] \\
C_1 \arrow[r] & C_1 \oplus C_2 \oplus C_{12}
\end{tikzcd}$$

One may form a morphism of partial 2-cubes

$$\begin{tikzcd}
 & * \arrow[d] & & & C_2 \arrow[d] \\
* \arrow[r] & C_{12} \arrow[r, phantom, "\Longrightarrow"] & & C_1 \arrow[r] & C_1 \oplus C_2 \oplus C_{12}
\end{tikzcd}$$

via naturality of 5.2.2 which, taking limits, constructs such a square. Taking fibers along the vertical maps, one has the identity map of $\Omega C_1 \oplus \Omega C_{12}$; the square is therefore cartesian by [1, 2.2]. $\square$

5.2.2 Slice towers and chromatic localizations

In this section, we use the slice filtration to work in the ∞-category $\mathrm{Fun}(\mathbb{Z}^{op}, \mathbf{Sp}^G)$ of filtered G-spectra (see [59, 1.2.2]). We refer to [90] for a treatment of the slice filtration in an ∞-categorical context. Let

$$\mathcal{T} : \mathbf{Sp}^G \to \mathrm{Fun}(\mathbb{Z}^{op}, \mathbf{Sp}^G)$$

be the functor which associates to a G-spectrum its slice tower, which may be obtained as in [59, 1.2.1.17]. We use the following notation in this context:

- $\widetilde{P}^k : \mathrm{Fun}(\mathbb{Z}^{op}, \mathbf{Sp}^{C_4}) \xrightarrow{\mathrm{ev}_k} \mathbf{Sp}^{C_4}$

- $\widetilde{P}^k_k = \mathrm{fib}(\widetilde{P}^k \to \widetilde{P}^{k-1})$

- $P^k = \tilde{P}^k \circ \mathcal{T}$

- $P^k_k = \tilde{P}^k_k \circ \mathcal{T}$

- $\mathrm{holim} : \mathrm{Fun}(\mathbb{Z}^{op}, \mathbf{Sp}^{C_4}) \to \mathbf{Sp}^{C_4}$ is the functor sending a tower to its homotopy limit.

We use extensively that limits and colimits are computed pointwise in functor categories. We begin with a useful lemma:

Lemma 5.2.6. *The functor $\mathcal{T} : \mathbf{Sp}^G \to \mathrm{Fun}(\mathbb{Z}^{op}, \mathbf{Sp}^G)$ commutes with filtered colimits.*

Proof. Let

$$X_1 \to X_2 \to X_3 \to \cdots$$

be an ind-system of G-spectra with colimit X. For all $k \in \mathbb{Z}$, we have a map of cofiber sequences

$$\begin{tikzcd}
\mathrm{colim}_i P_{k+1}(X_i) \arrow[r] \arrow[d] & \mathrm{colim}_i X_i \arrow[r] \arrow[d, "\simeq"] & \mathrm{colim}_i P^k(X_i) \arrow[d] \\
P_{k+1}(X) \arrow[r] & X \arrow[r] & P^k(X)
\end{tikzcd}$$

By the slice recognition principle [39, 4.16], the left and right arrows are equivalences provided that $\mathrm{colim}_i P_{k+1}(X_i)$ is slice $> k$ and $\mathrm{colim}_i P^k(X_i)$ is slice $\leq k$. The former follows from the fact that the subcategory

$$\tau_{>k} = \{Y \in \mathbf{Sp}^G : Y > k\}$$

is a localizing subcategory by definition, and the latter follows from the fact that slice spheres are compact.

The lemma now follows from the fact that equivalences in functor categories are detected pointwise. □

Lemma 5.2.6 allows us to easily determine the slice tower of the C_4-spectrum $BP^{((C_4))}_{i_1,\ldots,i_j}$. We define n so that $-n\rho_4 = |N_{C_2}^{C_4}(\overline{t_{i_1}}\cdots\overline{t_{i_j}})^{-1}|$, and

$$BP^{((C_4))}_{i_1,\ldots,i_j} = \mathrm{colim}\left(BP^{((C_4))} \xrightarrow{N_{C_2}^{C_4}(\overline{t_{i_1}}\cdots\overline{t_{i_j}})\cdot} \Sigma^{n\rho_4} BP^{((C_4))} \xrightarrow{N_{C_2}^{C_4}(\overline{t_{i_1}}\cdots\overline{t_{i_j}})\cdot} \cdots \right)$$

Proposition 5.2.7. *There is an equivalence of filtered C_4-spectra*

$$\mathcal{T}(BP^{((C_4))}_{i_1,\dots,i_j}) \simeq \mathrm{colim}_k \mathcal{T}(\Sigma^{kn\rho_4} BP^{((C_4))})$$

In particular, the localization

$$BP^{((C_4))} \to BP^{((C_4))}_{i_1,\dots,i_j}$$

induces the corresponding localization

$$H\underline{\mathbb{Z}}_{(2)} \wedge S^0[C_4 \cdot \overline{t_1}, C_4 \cdot \overline{t_2}, \dots] \to H\underline{\mathbb{Z}}_{(2)} \wedge S^0[C_4 \cdot \overline{t_1}, C_4 \cdot \overline{t_2}, \dots][C_4 \cdot (\overline{t_{i_1}} \cdots \overline{t_{i_j}})^{-1}]$$

on slice associated-graded. The notation is as in [39], where

$$S^0[C_4 \cdot \overline{t_1}, C_4 \cdot \overline{t_2}, \dots][C_4 \cdot (\overline{t_{i_1}} \cdots \overline{t_{i_j}})^{-1}] = N_{C_2}^{C_4}(S^0[\overline{t_1}, \overline{t_2}, \dots][(\overline{t_{i_1}} \cdots \overline{t_{i_j}})^{-1}])$$

Proof. The first claim follows immediately from Lemma 5.2.6. The description of the slice associated graded of $BP^{((C_4))}_{i_1,\dots,i_j}$ follows from the HHR slice theorem and [39, Corollary 4.25], which implies that

$$P^l_l(\Sigma^{kn\rho_4} BP^{((C_4))}) \simeq \Sigma^{kn\rho_4} P^{l-4kn}_{l-4kn} BP^{((C_4))}$$

□

Definition 5.2.8. We define the wedges of slice spheres $\widehat{W}^{i_1,\dots,i_j}_{2d}$ and $\widehat{W}_{2d}$ by

$$P^{2d}_{2d}(BP^{((C_4))}_{i_1,\dots,i_j}) = H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}^{i_1,\dots,i_j}_{2d}$$

$$P^{2d}_{2d}(BP^{((C_4))}) = H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d}$$

5.2.3 Proof that $BP^{((C_4))}$ is cofree

We introduce the chromatic n-cubes we need to prove that $BP^{((C_4))}$ is cofree and show they split as a summand that is constant in n and a cartesian n-cube built from disjoint split inclusions.

Definition 5.2.9. Consider the following hypercubes:

1. Let $\mathcal{H}_n$ be the cartesian n-cube so that for $\{i_1, \dots, i_j\} \in \mathcal{P}_0([n])$

$$\mathcal{H}_n(\{i_1, \dots, i_j\}) = BP^{((C_4))}_{i_1,\dots,i_j}$$

   One may form this cube inductively by working in the category of $MU^{((C_4))}_{(2)}$-modules and applying the functors $(-)[N(\overline{t_i})^{-1}]$. See [1, 3.1] for a similar construction.

2. Let $\mathcal{S}_{n,d}$ be the cartesian n-cube defined on $\mathcal{P}_0([n])$ by

$$\mathcal{S}_{n,d} : \mathcal{P}_0([n]) \xrightarrow{\mathcal{H}_n} \mathbf{Sp}^{C_4} \xrightarrow{P^{2d}_{2d}} \mathbf{Sp}^{C_4}$$

With notation as in Definition 5.2.8, we note that $\widehat{W}^{i_1,\dots,i_j}_{2d}$ has $\widehat{W}_{2d}$ as a split summand for any $\{i_1, \dots, i_j\}$, corresponding to the split inclusion

$$\pi^u_{2d}(BP^{((C_4))}) \hookrightarrow \pi^u_{2d}(BP^{((C_4))}_{i_1,\dots,i_j})$$

This splitting is natural in $\{i_1, \dots, i_j\}$, so we see that there is a splitting

$$\mathcal{S}_{n,d} \simeq (H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d}) \oplus \mathcal{X}_{n,d}$$

where $\mathcal{X}_{n,d}$ is a cartesian n-cube satisfying

$$\mathcal{X}_{n,d}(\{i_1, \dots, i_j\}) = H\underline{\mathbb{Z}}_{(2)} \wedge (\widehat{W}^{i_1,\dots,i_j}_{2d} / \widehat{W}_{2d})$$

We have the following connection to the generalities in 5.2.1:

Proposition 5.2.10. *The cube $\mathcal{X}_{n,d}$ is a cartesian n-cube built from disjoint split inclusions.*

Proof. $\mathcal{X}_{n,d}$ is cartesian by definition. The result - and the terminology - follows from the fact that for any $\{i_1, \dots, i_j\}$, the maps

$$\pi^u_*(BP^{((C_4))}_{i_k}) \hookrightarrow \pi^u_*(BP^{((C_4))}_{i_1,\dots,i_j})$$

are split inclusions, and after factoring out $\pi^u_*(BP^{((C_4))})$, the maps

$$\iota_k : \frac{\pi^u_*(BP^{((C_4))}_{i_k})}{\pi^u_*(BP^{((C_4))})} \hookrightarrow \frac{\pi^u_*(BP^{((C_4))}_{i_1,\dots,i_j})}{\pi^u_*(BP^{((C_4))})}$$

are split inclusions with the property that $\mathrm{im}(\iota_k) \cap \mathrm{im}(\iota_{k'}) = \{0\}$ for $k \neq k'$. Now the claim follows from the fact that

$$\frac{\pi_*^u(BP^{((C_4))}_{i_1,\ldots,i_j})}{\pi_*^u(BP^{((C_4))})} = \left(\bigoplus_{\substack{T < \{i_1,\ldots,i_j\} \\ T \in \mathcal{P}_0(n)}} \frac{\pi_*^u(BP^{((C_4))}_T)}{\pi_*^u(BP^{((C_4))})} \right) \oplus \frac{(t_{i_1} \cdots t_{i_j} \gamma(t_{i_1}) \cdots \gamma(t_{i_j}))^{<0} \pi_*^u(BP^{((C_4))})}{\pi_*^u(BP^{((C_4))})}$$

where the latter summand denotes the subgroup of $\frac{\pi_*^u(BP^{((C_4))}_{i_1,\ldots,i_j})}{\pi_*^u(BP^{((C_4))})}$ generated by monomials containing $(t_{i_1} \cdots t_{i_j} \gamma(t_{i_1}) \cdots \gamma(t_{i_j}))^{-k}$ for $k > 0$. □

The following is an immediate consequence of 5.2.5 and 5.2.10:

Corollary 5.2.11. *The map $\mathcal{S}_{n,d}(\varnothing) \to \mathcal{S}_{n-1,d}(\varnothing)$ can be identified with*

$$(H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d}) \oplus \mathcal{X}_{n,d}(\varnothing) \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} (H\underline{\mathbb{Z}}_{(2)} \wedge \widehat{W}_{2d}) \oplus \mathcal{X}_{n-1,d}(\varnothing)$$

The canonical map $BP^{((C_4))} \to BP^{((C_4))}_{i_1,\ldots,i_j}$, by universal property, determines compatible maps $BP^{((C_4))} \to \mathcal{H}_n(\varnothing)$ so that there is a map

$$BP^{((C_4))} \to \mathrm{holim}_n \mathcal{H}_n(\varnothing)$$

We will show this map is an equivalence, and this will complete the proof that $BP^{((C_4))}$ is cofree by the following.

Proposition 5.2.12. *The C_4-spectrum* $\mathrm{holim}_n \mathcal{H}_n(\varnothing)$ *is cofree.*

Proof. By Corollary 5.1.3, the category of cofree C_4-spectra is closed under limits, hence it suffices to show that each $\mathcal{H}_n(\varnothing)$ is cofree. There is by definition an equivalence

$$\mathcal{H}_n(\varnothing) \xrightarrow{\simeq} \mathrm{holim}_{T \in \mathcal{P}_0([n])} \mathcal{H}_n(T) = \mathrm{holim}_{\{i_1,\ldots,i_j\} \in \mathcal{P}_0([n])} BP^{((C_4))}_{i_1,\ldots,i_j}$$

so it suffices to show that each $BP^{((C_4))}_{i_1,\ldots,i_j}$ is cofree. $BP^{((C_4))}_{i_1,\ldots,i_j}$ is a module over

$$MU^{((C_4))}_{(2)}[N_{C_2}^{C_4}(\overline{t_{i_1}} \cdots \overline{t_{i_j}})^{-1}]$$

and so we may argue as in [39, Section 10]: we have that

$$\Phi^{C_4}(MU_{(2)}^{((C_4))}[N_{C_2}^{C_4}(\overline{t_{i_1}}\cdots\overline{t_{i_j}})^{-1}]) \simeq \Phi^{C_2}(MU_{(2)}^{((C_4))}[N_{C_2}^{C_4}(\overline{t_{i_1}}\cdots\overline{t_{i_j}})^{-1}]) \simeq *$$

as

$$\Phi^{C_4}(N_{C_2}^{C_4}(\overline{t_{i_1}})) = \Phi^{C_2}(\overline{t_{i_1}}) = 0$$

and similarly

$$\Phi^{C_2}(N_{C_2}^{C_4}(\overline{t_{i_1}})) = \Phi^{C_2}(i_{C_2}^{C_4}N_{C_2}^{C_4}(\overline{t_{i_1}})) = \Phi^{C_2}(\overline{t_{i_1}}\cdot\overline{\gamma(t_{i_1})}) = 0$$

□

To show that the map

$$BP^{((C_4))} \to \mathrm{holim}_n \mathcal{H}_n(\varnothing)$$

is an equivalence, we work instead in filtered C_4-spectra, where by functoriality we have a map

$$f : \mathcal{T}(BP^{((C_4))}) \to \mathrm{holim}_n \left(\mathrm{holim}_{\{i_1,\ldots,i_j\}\in\mathcal{P}_0([n])} \mathcal{T}(BP^{((C_4))}_{i_1,\ldots,i_j}) \right)$$

We will show that f is an equivalence, for which we need the following lemma:

Lemma 5.2.13. *Let $\mathcal{C}$ be a co-complete stable ∞-category. Suppose $\mathcal{T}_1, \mathcal{T}_2 \in \mathrm{Fun}(\mathbb{Z}^{op}, \mathcal{C})$ are such that*

$$\mathrm{colim}_k \tilde{P}^k(\mathcal{T}_1) \simeq \mathrm{colim}_k \tilde{P}^k(\mathcal{T}_2) \simeq *$$

If $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ has the property that $\tilde{P}^k_k(\phi)$ is an equivalence for all $k \in \mathbb{Z}$, then ϕ is an equivalence.

Proof. Let $\mathcal{T}_3 = \mathrm{cofib}(\phi : \mathcal{T}_1 \to \mathcal{T}_2)$, then it suffices to show that $\mathcal{T}_3 \simeq *$. We have that $\tilde{P}^k_k(\mathcal{T}_3) \simeq *$ for all $k \in \mathbb{Z}$ so that

$$\tilde{P}^k(\mathcal{T}_3) \to \tilde{P}^{k-1}(\mathcal{T}_3)$$

is an equivalence for all $k \in \mathbb{Z}$. Therefore $\mathcal{T}_3$ is equivalent to a constant tower, but since $\mathrm{colim}_k \tilde{P}^k(\mathcal{T}_3) \simeq *$, we must have $\mathcal{T}_3 \simeq *$. □

Theorem 5.2.14. *The C_4-spectrum $BP^{((C_4))}$ is cofree, independent of Lin's theorem.*

Proof. It suffices to show that f is an equivalence, as

$$\operatorname{holim}\left(\operatorname{holim}_{\{i_1,\dots,i_j\}\in\mathcal{P}_0([n])}\mathcal{T}(BP^{((C_4))}_{i_1,\dots,i_j})\right)\simeq\mathcal{H}_n(\varnothing)$$

and

$$\operatorname{holim}(\mathcal{T}(BP^{((C_4))}))\simeq BP^{((C_4))}$$

Note that

$$\begin{aligned}\widetilde{P}^k_k\left(\operatorname{holim}_{\{i_1,\dots,i_j\}\in\mathcal{P}_0([n])}\mathcal{T}(BP^{((C_4))}_{i_1,\dots,i_j})\right)&\simeq\operatorname{holim}_{\{i_1,\dots,i_j\}\in\mathcal{P}_0([n])}\widetilde{P}^k_k\left(\mathcal{T}(BP^{((C_4))}_{i_1,\dots,i_j})\right)\\&\simeq\begin{cases}* & k=2d-1\\ \mathcal{S}_{n,d}(\varnothing) & k=2d\end{cases}\end{aligned}$$

The map $\widetilde{P}^{2d}_{2d}(f)$ is then identified with the map

$$H\underline{\mathbb{Z}}_{(2)}\wedge\widehat{W}_{2d}\to\operatorname{holim}_n((H\underline{\mathbb{Z}}_{(2)}\wedge\widehat{W}_{2d})\oplus\mathcal{X}_{n,d})\simeq\operatorname{holim}_n(H\underline{\mathbb{Z}}_{(2)}\wedge\widehat{W}_{2d})\oplus\operatorname{holim}_n\mathcal{X}_{n,d}(\varnothing)$$

By Corollary 5.2.11, the lefthand summand is constant in n, and the righthand summand is pro-zero, hence the map is an equivalence.

To establish that f is an equivalence, by Lemma 5.2.13, it suffices now to show that

$$\operatorname{colim}_k\widetilde{P}^k\left(\operatorname{holim}_n\left(\operatorname{holim}_{\{i_1,\dots,i_j\}\in\mathcal{P}_0([n])}\mathcal{T}(BP^{((C_4))}_{i_1,\dots,i_j})\right)\right)\simeq *$$

i.e. that the filtration on $\operatorname{holim}_n\mathcal{H}_n(\varnothing)$ *strongly* converges. Note that by ([39], 4.42), if $X\in\mathbf{Sp}^{C_4}$, then $\underline{\pi}_l(P^kX)=0$ for $l>\lfloor(k+1)/4\rfloor$ when $k<0$ and for $k>l$ when $k\geq 0$. Taking limits, it follows that

$$\underline{\pi}_l\left(\widetilde{P}^k\left(\operatorname{holim}_{\{i_1,\dots,i_j\}\in\mathcal{P}_0([n])}\mathcal{T}(BP^{((C_4))}_{i_1,\dots,i_j})\right)\right)=0$$

in the same range, and so

$$\underline{\pi}_l\left(\widetilde{P}^k\left(\operatorname{holim}_n\left(\operatorname{holim}_{\{i_1,\dots,i_j\}\in\mathcal{P}_0([n])}\mathcal{T}(BP^{((C_4))}_{i_1,\dots,i_j})\right)\right)\right)=0$$

in the same range by the Milnor sequence. It follows that, for any l, taking the colimit as $k \to -\infty$ of $\underline{\pi}_l$ gives zero. $\square$

Remark 5.2.15. This result recovers the Hu-Kriz result that $BP_{\mathbb{R}}$ is cofree: since $BP^{((C_4))}$ is cofree, $i_{C_2}^{C_4} BP^{((C_4))} = BP_{\mathbb{R}} \wedge BP_{\mathbb{R}}$ is cofree, hence so is the retract $BP_{\mathbb{R}}$. Alternatively, as discussed in the introduction, one may argue similarly to 5.2.14 to show that $BP_{\mathbb{R}}$ is cofree, and the result in this case is due to Mike Hill.

Chapter 6

CHROMATIC MEASURE AND STACKS ASSOCIATED TO REAL JOHNSON-WILSON THEORIES

In this chapter, we bring Real-oriented homotopy theory into the world of stacks by applying a construction of Hopkins' to various fixed point spectra, which we review in Section 6.1. In Section 6.2, we use this construction to define an invariant of a ring spectrum E, a natural number that we call *chromatic measure*. Roughly, this measures how far E is from being complex orientable. We demonstrate that this measure is computable given a suitable description of the associated stack $\mathcal{M}_E$.

As an application, we focus in particular on the Real Johnson-Wilson theories $E_{\mathbb{R}}(n)$ and their fixed points $ER(n)$. We compute the chromatic measure of $ER(n)$ for all n and give several modular descriptions in Section 6.3 of the associated stack $\mathcal{M}_{ER(n)}$. This recovers and generalizes results of Hopkins on $ER(1) \simeq KO$. We give similar results for related Morava E-theories and comment in Section 6.4 on future directions of this work, aimed at generalizing such results to connective versions of the $ER(n)$'s.

We also give a way to compute the chromatic measure of a higher real K-theory spectrum, EO_n, in terms of the valuation ν on the endomorphism ring $\mathrm{End}(\mathbb{G})$ of the corresponding formal group of height n, and perform this calculation explicitly for the standard C_p subgroup of the n-th Morava stabilizer group when $p-1|n$.

Throughout this chapter, we continue to work in the flat topology on affine schemes, and we fix our ground ring to be $k = \mathbb{Z}$, or when working p-locally, $k = \mathbb{Z}_{(p)}$.

6.1 Hopkins' *From Spectra to Stacks*

In this section, we recall some notions from Mike Hopkins' lecture *From Spectra to Stacks* in [22]. These lecture notes have been particularly inspiring to the author, and we encourage anyone interested in chromatic homotopy to study them seriously. In particular, we sample only a few concepts from these notes, as needed for our results in this chapter. There are a number of other ideas in these notes of which the interested reader should be aware, and we cannot hope to improve upon their exposition therein. Our notion of chromatic measure below was certainly implicit in these notes; we have simply taken care to formally define it. We begin with Hopkins' definition of *associated stack*:

Definition 6.1.1. Suppose E is a homotopy-commutative ring spectrum with the property that the commutative ring MU_*E is concentrated in even degrees. The pair

$$(MU_*E, MU_*(MU \wedge E))$$

determines a Hopf algebroid, and we define the *stack associated to E* by

$$\mathcal{M}_E := \mathcal{M}_{(MU_*E, MU_*(MU \wedge E))}$$

as in Definition 2.1.39.

Remark 6.1.2. We include the evenness assumption here so that MU_*E is an honest commutative ring. It is of course possible to work with stacks on graded-commutative rings, but we do not pursue that here. Note also that the Kunneth map

$$MU_*MU \otimes_{MU_*} MU_*E \to MU_*(MU \wedge E)$$

is an isomorphism because MU_*MU is flat over MU_*. Thus equivalently, we have

$$\mathcal{M}_E = \mathcal{M}_{(MU_*E, MU_*MU \otimes_{MU_*} MU_*E)}$$

Under this identification, η_R is identified with the coaction map on the (MU_*, MU_*MU)-comodule MU_*E.

Remark 6.1.3. From the point of view of computations in chromatic homotopy, the cohomology of the sheaf $\mathcal{O}_{\mathcal{M}_E}$ on $\mathcal{M}_E$ is the E_2-page of the Adams-Novikov spectral sequence for E. Thus, the stackiness of $\mathcal{M}_E$ is in some sense measuring both the failure of E to be complex-orientable and the complexity of the ANSS for E. We make this somewhat precise in the next section.

The unit $\mathbb{S} \to E$ of the ring spectrum E provides a map of Hopf algebroids $(MU_*, MU_*MU) \to (MU_*E, MU_*(MU \wedge E))$ which determines a morphism of stacks

$$\phi_E : \mathcal{M}_E \to \mathcal{M}_{FG}(1)$$

In light of Remark 6.1.2, we may view this construction as a relative Spec construction: MU_*E is an (MU_*, MU_*MU)-comodule algebra - i.e. a quasicoherent sheaf of algebras $\mathcal{F}_E$ on $\mathcal{M}_{FG}(1)$ - such that

$$(\phi_E)_*\mathcal{O}_{\mathcal{M}_E} \cong \mathcal{F}_E$$

We have more generally:

Definition 6.1.4. Let (A, Γ) be a Hopf algebroid, and M a (left)-comodule algebra over (A, Γ) via

$$\psi : M \to \Gamma \otimes_A M$$

M determines a Hopf algebroid $(M, \Gamma \otimes_A M)$ via the following structure maps

- $\eta_L : M \to \Gamma \otimes_A M$ is given by $m \mapsto 1 \otimes m$.
- $\eta_R = \psi$.
- $\epsilon : \Gamma \otimes_A M \to M$ is defined by the commutative diagram

$$\begin{tikzcd} \Gamma \otimes_A M \arrow[r, "\epsilon"] \arrow[d, "\epsilon' \otimes 1"'] & M \\ A \otimes_A M \arrow[ur, "\cong"'] & \end{tikzcd}$$

where $\epsilon' : \Gamma \to A$ is the identity morphism in (A, Γ).

- Δ is defined by the commutative diagram

$$\begin{tikzcd}
\Gamma \otimes_A M \arrow[r, "\Delta"] \arrow[d, "\Delta' \otimes \psi"'] & (\Gamma \otimes_A M) \otimes_M (\Gamma \otimes_A M) \arrow[r, "\cong"] & \Gamma \otimes_A \Gamma \otimes_A M \\
\Gamma \otimes_A \Gamma \otimes_A \Gamma \otimes_A M \arrow[urr, "1 \otimes \mu \otimes 1"'] & &
\end{tikzcd}$$

where $\Delta' : \Gamma \to \Gamma \otimes_A \Gamma$ is the composition morphism in (A, Γ).

- $c : \Gamma \otimes_A M \to \Gamma \otimes_A M$ is defined by the commutative diagram

$$\begin{tikzcd}
\Gamma \otimes_A M \arrow[r, "c"] \arrow[d, "c' \otimes 1"'] & \Gamma \otimes_A M \\
\Gamma \otimes_A M \arrow[r, "1 \otimes \psi"] & \Gamma \otimes_A \Gamma \otimes_A M \arrow[u, "\mu \otimes 1"']
\end{tikzcd}$$

where $c' : \Gamma \to \Gamma$ is the inversion morphism in (A, Γ).

We will make use of the following fact to identify $\mathcal{M}_E$ when E is complex orientable.

Lemma 6.1.5. *In Definition 6.1.4, let $M = \Gamma$ be the left (A, Γ)-comodule algebra with comodule structure map*

$$\psi = \Delta : \Gamma \to \Gamma \otimes_A \Gamma$$

The identity morphism $\epsilon : \Gamma \to A$ induces an equivalence of stacks

$$\mathrm{Spec}(A) \to \mathcal{M}_{(\Gamma, \Gamma \otimes_A \Gamma)}$$

Proof. We will prove that the map

$$\mathrm{Spec}(A) \to \mathcal{M}^{pre}_{(\Gamma, \Gamma \otimes_A \Gamma)}$$

is an equivalence of prestacks; the result follows by applying stackification. We claim first that the prestack $\mathcal{M}^{pre}_{(\Gamma, \Gamma \otimes_A \Gamma)}$ is discrete, in the sense of Definition 2.1.19. This follows from the claim that the sequence

$$\Gamma \underset{\Delta}{\overset{g \mapsto 1 \otimes g}{\rightrightarrows}} \Gamma \otimes_A \Gamma \xrightarrow{\epsilon \otimes 1} \Gamma$$

is a coequalizer sequence. This is equivalent to the claim that, for every commutative ring R, the sequence

$$\mathrm{Spec}(\Gamma)(R) \longrightarrow \mathrm{Spec}(\Gamma)(R) \times_{\mathrm{Spec}(A)(R)} \mathrm{Spec}(\Gamma)(R) \underset{\pi_2}{\overset{\text{composition}}{\rightrightarrows}} \mathrm{Spec}(\Gamma)(R)$$

is an equalizer sequence in **Sets**. $(\mathrm{Spec}(A)(R), \mathrm{Spec}(\Gamma)(R))$ forms a groupoid $\mathcal{C}_R$, and the first map in the above sequence sends

$$(x \xrightarrow{f} y) \mapsto (\mathrm{id}_y, x \xrightarrow{f} y)$$

where $x \xrightarrow{f} y$ is a morphism in $\mathcal{C}_R$. The claim then follows from the obvious bijection

$$\{(y \xrightarrow{g} z, x \xrightarrow{f} y) : g \circ f = f\} \cong \{(\mathrm{id}_y, x \xrightarrow{f} y)\}$$

It suffices now to show that for each commutative ring R, the map

$$\mathrm{Spec}(A)(R) \to \mathcal{M}^{pre}(R)$$

is essentially surjective. It suffices to show this for the universal object $\mathrm{id} : \Gamma \to \Gamma$ in the groupoid $\mathcal{M}^{pre}(\Gamma)$, which follows from the commutativity of the diagram

$$\begin{tikzcd}
\Gamma \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{id}"] & & \\
& \Gamma \otimes_A \Gamma \arrow[r, "1\otimes\epsilon"] & \Gamma \\
\Gamma \arrow[ur, "g\mapsto 1\otimes g"] \arrow[r, "\epsilon"'] & A \arrow[ur, "\eta_R"'] &
\end{tikzcd}$$

recalling the diagram from Remark 2.1.42. □

Proposition 6.1.6. *Let E and F be homotopy commutative ring spectra such that MU_*E and MU_*F are even.*

1. *If E is complex-orientable, $\mathcal{M}_E \simeq \mathrm{Spec}(E_*)$, and ϕ_E is the map classifying the formal group over E_*.*

2. Suppose that MU_*F is a flat MU_*-module, then one has a pullback square

$$\begin{tikzcd} \mathcal{M}_{E\wedge F} \arrow[r] \arrow[d] & \mathcal{M}_F \arrow[d, "\phi_F"] \\ \mathcal{M}_E \arrow[r, "\phi_E"'] & \mathcal{M}_{FG}(1) \end{tikzcd}$$

3. ϕ_E is an affine morphism.

Proof. For (1), fix a complex orientation $MU \to E$, and note first that since E is complex orientable, $MU_*E \cong E_*[b_i]$ with $|b_i| = 2i$ by Proposition 2.2.8, so E_* is even if and only if MU_*E is even. Lemma 6.1.5 gives an equivalence

$$\mathrm{Spec}(MU_*) \simeq \mathcal{M}_{(MU_*MU, MU_*MU \otimes_{MU_*} MU_*MU)} = \mathcal{M}_{MU}$$

The complex orientation of E determines a ring map $MU_* \to E_*$. Base changing the above equivalence along this map we have

$$\begin{aligned} \mathrm{Spec}(E_*) &\simeq \mathrm{Spec}(MU_*) \times_{\mathrm{Spec}(MU_*)} \mathrm{Spec}(E_*) \\ &\simeq \left(\mathcal{M}_{(MU_*MU, MU_*MU \otimes_{MU_*} MU_*MU)}\right) \times_{\mathrm{Spec}(MU_*)} \mathrm{Spec}(E_*) \\ &\simeq \mathcal{M}_{(MU_*E, MU_*MU \otimes_{MU_*} MU_*E)} \\ &= \mathcal{M}_E \end{aligned}$$

using the isomorphism of Proposition 2.2.8.

For (2), we first prove the claim in the case $F = MU$. $MU_*MU \cong MU_*[b_i]$ is flat over MU_*, so the hypotheses apply. In this case, one checks directly that the map

$$(MU_*, MU_*MU) \hookrightarrow (MU_*MU, MU_*MU \otimes_{MU_*} MU_*MU)$$

induces a fibration of stacks $\mathcal{M}_{MU} \to \mathcal{M}_{FG}(1)$ (note that neither needs stackification by Theorem 2.1.64), in the sense of Remark 2.1.29. We may therefore compute the homotopy pullback

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{M}_{MU} \arrow[d, "\phi_F"] \\ \mathcal{M}_E \arrow[r, "\phi_E"'] & \mathcal{M}_{FG}(1) \end{tikzcd}$$

as the *strict* pullback. Moreover, we may compute this homotopy pullback *before* stackification by Remark 2.1.28. It is straightforward to show then that

$$\mathcal{P} \simeq \mathcal{M}_{(MU_*MU \otimes_{MU_*} MU_*E, MU_*MU \otimes_{MU_*} MU_*MU \otimes_{MU_*} MU_*E)}$$

Finally, using the Kunneth map and that MU_*MU is flat over MU_*, we have an isomorphism of Hopf algebroids

$$\begin{aligned}
&(MU_*MU \otimes_{MU_*} MU_*E, MU_*MU \otimes_{MU_*} MU_*MU \otimes_{MU_*} MU_*E) \\
&\xrightarrow{\cong} (MU_*(MU \wedge E), MU_*(MU \wedge MU \wedge E))
\end{aligned}$$

so that $\mathcal{P} \simeq \mathcal{M}_{E \wedge MU}$.

In the general case, we have a diagram as follows

$$\begin{tikzcd}
& \mathrm{Spec}(MU_*E) \arrow[rr] \arrow[dd] & & \mathrm{Spec}(MU_*) \arrow[dd] \\
\mathrm{Spec}(MU_*E \otimes_{MU_*} MU_*F) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Spec}(MU_*F) \arrow[ur] \arrow[dd] & \\
& \mathcal{M}_E \arrow[rr] & & \mathcal{M}_{FG}(1) \\
\mathcal{P} \arrow[ur] \arrow[rr] & & \mathcal{M}_F \arrow[ur] &
\end{tikzcd}$$

where each square is a pullback. We have used the MU case to identify the pullbacks in the back and righthand faces, and we have used (1) to identify $\mathcal{M}_{E \wedge MU}, \mathcal{M}_{F \wedge MU}$, and $\mathcal{M}_{MU}$ with the corresponding affine schemes. We wish to show that $\mathcal{P} \simeq \mathcal{M}_{E \wedge F}$; $\mathcal{P}$ is locally presentable, as a pullback of locally presentable stacks, and the above diagram implies that

$$\mathrm{Spec}(MU_*E \otimes_{MU_*} MU_*F) \to \mathcal{P}$$

is a faithfully flat cover. Set

- $R \coloneqq MU_*$
- $R_{MU} \coloneqq MU_*MU$
- $R_E \coloneqq MU_*E$

- $R_F \coloneqq MU_*F$

and consider then the following diagram of pullback squares

$$\begin{tikzcd}
& \mathrm{Spec}(R_{MU} \otimes_R R_F) \arrow[rr] \arrow[dd] & & \mathrm{Spec}(R_F) \arrow[dd] \\
\mathrm{Spec}(R_{MU} \otimes_R R_E \otimes_R R_F) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Spec}(R_E \otimes_R R_F) \arrow[ur] \arrow[dd] & \\
& \mathrm{Spec}(R_F) \arrow[rr] & & \mathcal{M}_F \\
\mathrm{Spec}(R_E \otimes_R R_F) \arrow[ur] \arrow[rr] & & \mathcal{P} \arrow[ur] &
\end{tikzcd}$$

This determines an equivalence

$$\mathcal{M}_{(MU_*E \otimes_{MU_*} MU_*F, MU_*MU \otimes_{MU_*} MU_*E \otimes_{MU_*} MU_*F)} \to \mathcal{P}$$

by Theorem 2.1.44. Using the Kunneth map and flatness of MU_*F, as before, one has an equivalence

$$\mathcal{M}_{E \wedge F} \simeq \mathcal{M}_{(MU_*E \otimes_{MU_*} MU_*F, MU_*MU \otimes_{MU_*} MU_*E \otimes_{MU_*} MU_*F)}$$

For (3), it suffices to show the pullback

$$\begin{tikzcd}
\mathcal{P} \arrow[r] \arrow[d] & \mathrm{Spec}(MU_*) \arrow[d] \\
\mathcal{M}_E \arrow[r, "\phi_E"] & \mathcal{M}_{FG}(1)
\end{tikzcd}$$

$\mathcal{P}$ is affine, since $\mathrm{Spec}(MU_*) \to \mathcal{M}_{FG}(1)$ is a faithfully flat cover. This follows from (1) and (2) as $\mathcal{P} \simeq \mathcal{M}_{E \wedge MU} \simeq \mathrm{Spec}(E_*MU)$. $\square$

6.2 Chromatic measure

For E a homotopy commutative ring spectrum, we introduce an invariant of E called *chromatic measure* which, roughly, measures the failure of E to be complex-orientable. When MU_*E is even - so that we may associate to E the stack $\mathcal{M}_E$ - this invariant becomes quite computable given a modular description $\mathcal{M}_E$. In this section, we define this invariant and compute it for $E = ER(n)$, the fixed points of $E_{\mathbb{R}}(n)$, the n-th Real Johnson-Wilson theory, for all $n \geq 0$. We also demonstrate how to compute $\Phi(EO_n)$ for EO_n a higher real K-theory, in terms of the valuation ν on the endomorphism ring $\mathrm{End}(\mathbb{G})$ of the corresponding formal group of height n.

6.2.1 The $X(n)$-spectra and $\mathcal{M}_{FG}(n)$

We begin by recalling Ravenel's family of Thom spectra $X(n)$. By Bott periodicity, one has an equivalence $\Omega SU \simeq BU$, and one defines

$$X(n) \coloneqq \mathrm{Thom}(\Omega SU(n) \to \Omega SU \simeq BU)$$

They are E_2 algebras in $\mathbf{Sp}$; note that $X(1) \simeq S^0$ and $X(\infty) \simeq MU$. Therefore, MU admits an $X(n)$-orientation for all n, and one has Thom isomorphisms

$$MU_*X(n) \cong MU_*[b_1, \dots, b_{n-1}]$$

See [79, Section 9] and [22] for more details. The $X(n)$'s are especially crucial to Hopkins' associated stacks $\mathcal{M}_E$ because of the following.

Proposition 6.2.1. *Let $\mathcal{M}_{FG}(m)$ be the moduli stack of formal groups together with an m-jet, as in Definition 2.1.63. There is an equivalence of stacks $\mathcal{M}_{FG}(m) \simeq \mathcal{M}_{X(m)}$.*

Proof. By the Thom isomorphism, one has an isomorphism of Hopf algebroids

$$(MU_*X(m), MU_*(MU \wedge X(m))) \cong (MU_*[b_1, \dots, b_{m-1}], MU_*MU[b_1, \dots, b_{m-1}])$$

The map of Hopf algebroids

$$(MU_*X(m), MU_*(MU \wedge X(m))) \to (MU_*MU, MU_*(MU \wedge MU))$$

is an inclusion, so we may compute $\eta_R(b_i)$ in the latter. In MU_*MU, the b_i are by definition the coefficients of the canonical strict isomorphism $\eta_L^*F \to \eta_R^*F$ (see Lemma 2.2.4) where F is the universal formal group law over MU_* and

$$\eta_L : MU \simeq S^0 \wedge MU \to MU \wedge MU$$

$$\eta_R : MU \simeq MU \wedge S^0 \to MU \wedge MU$$

Note that a map

$$\phi : MU_*MU \otimes_{MU_*} MU_*MU \to R$$

corresponds to a sequence

$$F_1 \xleftarrow{g} F_2 \xleftarrow{f} F_3$$

of isomorphisms of formal group laws over R, where F_i is the pushforward of F_{MU} along the the map $MU \to MU \wedge MU \wedge MU$ given by mapping MU to the i-th smash factor. Since η_R in the Hopf algebroid $(MU_*MU, MU_*(MU \wedge MU))$ is induced by the map

$$MU \wedge MU \simeq MU \wedge S^0 \wedge MU \to MU \wedge MU \wedge MU$$

we have that $\phi(b_i)$ are the coefficients of f, and $\phi(\eta_R(b_i))$ are the coefficients of $g \circ f$. Therefore, if $\eta_L(b_i) = \eta_R(b_i)$ for $i \le m-1$, then $g(x) \equiv x \mod x^{m+1}$. It follows that the map

$$\mathcal{M}_{X(m)} \to \mathcal{M}_{FG}(1)$$

factors through an equivalence onto the substack $\mathcal{M}_{FG}(m)$. □

Proposition 6.2.2. *The map $\mathcal{M}_{FG}(m) \to \mathcal{M}_{FG}(1)$ is a faithfully flat cover with the property that*

$$\begin{tikzcd}
\mathcal{M}_{E \wedge X(m)} \arrow[r] \arrow[d] & \mathcal{M}_{FG}(m) \arrow[d] \\
\mathcal{M}_E \arrow[r] & \mathcal{M}_{FG}(1)
\end{tikzcd}$$

is a pullback.

Proof. This follows from the fact that $MU_*X(m) \cong MU_*[b_1, \dots, b_{m-1}]$ is faithfully flat over MU_* along with Proposition 6.2.1 □

Definition 6.2.3. For E a homotopy commutative ring spectrum, we define the *chromatic measure* of E to be the integer

$$\Phi(E) := \min\{n \ge 0 : X(n) \wedge E \text{ is complex-orientable}\}$$

Example 6.2.4. 1. Since $X(1) \simeq S^0$, $\Phi(E) = 1$ if and only if E is complex orientable.

2. In [22], Hopkins shows that $\Phi(KO) = \Phi(ko) = 2$ and $\Phi(tmf) = 4$.

Proposition 6.2.5. *Suppose E is a homotopy commutative ring spectrum with the property that MU_*E is even. If the pullback*

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{M}_{FG}(n) \\ \downarrow & & \downarrow \\ \mathcal{M}_E & \longrightarrow & \mathcal{M}_{FG}(1) \end{array}$$

$\mathcal{P}$ is affine, then $\Phi(E) \le n$. Conversely, if E is connective and $\Phi(E) \le n$, then $\mathcal{P}$ is affine.

Proof. By Remark 6.2.2, one has an equivalence

$$\mathcal{P} \simeq \mathcal{M}_{E \wedge X(n)}$$

If $\Phi(E) \le n$, then $E \wedge X(n)$ is complex orientable and hence $\mathcal{M}_{E\wedge X(n)} \simeq \mathrm{Spec}(E_*X(n))$. Conversely if $\mathcal{M}_{E\wedge X(n)}$ is affine, then

$$\mathrm{Ext}^{s,t}_{(MU_*, MU_*MU)}(MU_*, MU_*(E \wedge X(n))) \cong H^s(\mathcal{M}_{E\wedge X(n)}; \mathcal{O}_{\mathcal{M}_{E\wedge X(n)}}) = 0$$

for $s > 0$. The Adams-Novikov spectral sequence - which converges at E since E is assumed connective - is thus concentrated in the 0-line, and we have an iso

$$E_*X(n) \cong \mathrm{Hom}_{(MU_*, MU_*MU)}(MU_*, MU_*(E \wedge X(n)))$$

The latter graded abelian group is concentrated in even degrees because there is a Kunneth isomorphism

$$MU_*(E \wedge X(n)) \cong MU_*E \otimes_{MU_*} MU_*X(n)$$

as $MU_*X(n)$ is flat over MU_*. $\square$

6.2.2 *The chromatic measure of Real Johnson-Wilson theories and EO_n*

We are interested in a particular class of examples of associated stacks $\mathcal{M}_E$ via fixed points of genuine equivariant spectra. Suppose $E \in \mathbf{Sp}^G$ is a ring G-spectrum such that the underlying spectrum i_e^*E is complex-oriented. In most examples of interest, the fixed point spectrum E^G is not complex-oriented, hence we may try and use Hopkins' associated stack construction to bring E^G into the chromatic picture, and in particular we may ask if the moduli problem $\mathcal{M}_{E^G}$ is related to the formal group over $\pi_*(i_e^*E)$. One always has a factorization

$$\begin{tikzcd} \mathrm{Spec}(\pi_*(i_e^*E)) \arrow[r] \arrow[d] & \mathcal{M}_{FG}(1) \\ \mathrm{Spec}(\pi_*(i_e^*E))/G \arrow[ur, dashed] & \end{tikzcd}$$

This is because if $f : MU \to E$ is a complex orientation, and $g : E \to E$ is a ring automorphism, there is a canonical strict isomorphism from the formal group law classified by f to the formal group law classified by $g \circ f$ since each arise from complex orientations of F, and thus determine a coordinate on the same underlying formal group via Lemma 2.2.4. If the dashed arrow is affine, we will see it is often the case that

$$\mathcal{M}_{E^G} \simeq \mathrm{Spec}(\pi_*(i_e^*E))/G$$

Lemma 6.2.6. *Let A be a ring, R an A-algebra, and G a finite group acting on R in the category of A-algebras.*

1. *Suppose that G acts freely on the functor of points $\mathrm{Hom}_{\mathbf{CAlg}_A}(R, -)$ in the sense that for any nonzero A-algebra S,*

$$\mathrm{Hom}_{\mathbf{CAlg}_A}(R, S)$$

is a free G-set. Then $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ is a finite etale G-torsor in $\mathbf{Aff}_{/\mathrm{Spec}(A)}$, *and*

$$R \otimes_{R^G} R \to \prod_{g \in G} R$$

$$x \otimes y \mapsto (x \cdot g(y))_{g \in G}$$

is an isomorphism of left R-algebras

2. $\mathrm{Spec}(R^G) \simeq \mathrm{Spec}(R)/G$.

3. *If $\mathcal{M}$ is a discrete stack with an affine morphism $\mathcal{M} \to \mathrm{Spec}(R)/G$, then $\mathcal{M}$ is an affine scheme.*

Proof. (1) is [49, Theorem A7.1.1]. For (2), note that the G-torsor $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ is classified by the map $\mathrm{Spec}(R^G) \to \mathrm{Spec}(R)/G$ given by the descent datum on the cover $\mathrm{Spec}(R) \to \mathrm{Spec}(R^G)$ consisting of the identity map $R \to R$ with the isomorphism on the intersection given by the map

$$\prod_{g \in G} R \to R \otimes_{R^G} R$$

inverse to the isomorphism in (1). This shows, in particular, that the following diagram commutes and is a pullback

$$\begin{tikzcd}
\mathrm{Spec}(R) \arrow[r, "="] \arrow[d] & \mathrm{Spec}(R) \arrow[d] \\
\mathrm{Spec}(R^G) \arrow[r] & \mathrm{Spec}(R)/G
\end{tikzcd}$$

where $\mathrm{Spec}(R) \to \mathrm{Spec}(R)/G$ is the canonical G-torsor. We thus have the following commutative diagram

$$\begin{tikzcd}
& \mathrm{Spec}(\prod_{g \in G} R) \arrow[rr] \arrow[dd] & & \mathrm{Spec}(R) \arrow[dd] \\
\mathrm{Spec}(R \otimes_{R^G} R) \arrow[ur, "\cong"] \arrow[rr] \arrow[dd] & & \mathrm{Spec}(R) \arrow[ur, "="] \arrow[dd] & \\
& \mathrm{Spec}(R) \arrow[rr] & & \mathrm{Spec}(R)/G \\
\mathrm{Spec}(R) \arrow[ur, "="] \arrow[rr] & & \mathrm{Spec}(R^G) \arrow[ur] &
\end{tikzcd}$$

where the front and back faces are pullback squares. The fact that the upper left arrow is an iso implies that we have a diagram

$$\begin{tikzcd} \mathcal{M}_{(R, R\otimes_{R^G} R)} \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & \mathrm{Spec}(R^G) \arrow[d] \\ & \mathrm{Spec}(R)/G \end{tikzcd}$$

by Theorem 2.1.44, and thus $\mathrm{Spec}(R^G) \to \mathrm{Spec}(R)/G$ is an equivalence. For (3), since the morphism is affine, we have a pullback square

$$\begin{tikzcd} \mathrm{Spec}(\tilde{R}) \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathrm{Spec}(R) \arrow[r] & \mathrm{Spec}(R)/G \end{tikzcd}$$

for some ring $\tilde{R}$. The top horizontal map in the above diagram is a G-torsor and hence $\mathcal{M} \simeq \mathrm{Spec}(\tilde{R})/G$. It therefore suffices to show that G acts on $\tilde{R}$ under the conditions of (1), which follow since $\mathcal{M}$ is discrete: a G-set X has a free action if and only if the action groupoid of X is discrete. $\square$

We now turn our attention to the application mentioned above. Let $E_{\mathbb{R}}(n)$ denote the n-th Real Johnson-Wilson theory, as in Example 3.2.11 and let $ER(n) := (E_{\mathbb{R}}(n))^{C_2} \simeq (E_{\mathbb{R}}(n))^{hC_2}$. As before, we have a factorization:

$$\begin{tikzcd} \mathrm{Spec}(E(n)_*) \arrow[r] \arrow[d] & \mathcal{M}_{FG}(1) \\ \mathrm{Spec}(E(n)_*)/C_2 \arrow[ur, dashed, "p"'] & \end{tikzcd}$$

Proposition 6.2.7. *The morphism p is affine.*

Proof. Since $\phi_{X(m)}$ is a faithfully flat cover, it suffices to show that for some m, the stack

$$\mathcal{N} := \mathrm{Spec}(E(n)_*)/C_2 \times_{\mathcal{M}_{FG}(1)} \mathcal{M}_{FG}(m)$$

is an affine scheme. By Lemma 6.2.6 (3), it suffices to show $\mathcal{N}$ is discrete, and since the stackification of a discrete prestack is a discrete stack, it suffices to show that

$$\mathcal{N}^{pre} := (\mathrm{Spec}(E(n)_*)/C_2)^{pre} \times_{\mathcal{M}_{FG}(1)} \mathcal{M}_{FG}(m)$$

is a discrete prestack, where $(\mathrm{Spec}(E(n)_*)/C_2)^{pre}(R)$ is the action groupoid of the C_2-set $\mathrm{Hom}_{\mathbf{CAlg}}(E(n)_*, R)$, as in Example 2.1.24.

Referring to Definition 2.1.25, an element of $\mathcal{N}^{pre}(R)$ is a triple $(f, \mathbb{G}, \phi)$ where

- $f : E(n)_* \to R$ is a ring map
- $\mathbb{G}$ is a formal group law over R
- $\phi : f^* F_{E(n)} \to \mathbb{G}$ is a strict isomorphism

Set $F := f^* F_{E(n)}$. An automorphism of the triple $(f, \mathbb{G}, \phi)$ consists of

- An automorphism of f in the action groupoid, i.e. $g \in C_2$ such that $g \circ f = f$
- An automorphism $\psi : \mathbb{G} \to \mathbb{G}$ that is the identity mod x^{m+1} such that the following diagram in $\mathcal{M}_{FG}(1)(R)$ commutes.

$$\begin{tikzcd} F \arrow[r, "p(g)"] \arrow[d, "\phi"'] & F \arrow[d, "\phi"] \\ \mathbb{G} \arrow[r, "\psi"] & \mathbb{G} \end{tikzcd}$$

If the element $g \in C_2$ is the identity, then $p(g) = \mathrm{id}_F$ and the above diagram implies that ψ is the identity of $\mathbb{G}$, in which case we have the identity automorphism of $(f, \mathbb{G}, \phi)$. If $g = \gamma$ is the generator of C_2, then $p(g) = -[-1]_F$, as the action map

$$\gamma : i_e^* MU_{\mathbb{R}} \to i_e^* MU_{\mathbb{R}}$$

induces the map on homotopy groups classifying the conjugate of the universal formal group law F_{univ} by $-[-1]_{F_{univ}}(x)$ (see [39, Example 11.19]).

We now set $m = 2^n$, and note that since $f(v_n)$ is a unit in R, and

$$f(v_n) = f(\gamma(v_n)) = f(-v_n) = -f(v_n)$$

we have that $2f(v_n) = 0$, so R is an $\mathbb{F}_2$-algebra, and in particular $-[-1]_F(x) = [-1]_F(x)$. The above diagram therefore implies that

$$[-1]_F(x) \equiv x \mod x^{2^n+1}$$

so that

$$0 = F(x, [-1]_F(x)) \equiv F(x,x) \mod x^{2^n+1}$$

but the right hand side is $[2]_F(x)$, and F has height $\leq n$, a contradiction for a nonzero ring R. □

Remark 6.2.8. The same proof works with $\mathrm{Spec}(L)$ in place of $\mathcal{M}_{FG}(2^n)$, but this proof will tell us how the $ER(n)$'s interact with the $X(m)$'s.

These purely stack-theoretic observations have consequences for $ER(n)$ because the E_2 page of the HFPSS computing $ER(n)_*X$ is the cohomology of a sheaf on $\mathrm{Spec}(E(n)_*)/C_2$. We first need a couple of lemmas to help us access the cohomology of this sheaf.

Lemma 6.2.9. *Let E be a cofree C_2-spectrum, and X a spectrum, then $(E \wedge i_*X)^{hC_2} \simeq E^{hC_2} \wedge X$ if and only if $E \wedge i_*X$ is also cofree. This is always true if E is a module over a C_2-ring spectrum R such that $\Phi^{C_2}(R) \simeq *$.*

Proof. This is immediate from the fact that $E^{C_2} \simeq E^{hC_2}$ and $(-)^{C_2}$ commutes with colimits. For the second claim, note that $\Phi^{C_2}(E \wedge i_*X)$ and $(E \wedge i_*X)^{tC_2}$ are both modules over $\Phi^{C_2}(R)$. □

Note that this lemma applies when $E = E_{\mathbb{R}}(n)$ or $E = E_n$ with its central C_2 action from the Morava stabilizer group as in 3.2.12, because these are both $MU_{\mathbb{R}}[\overline{v_n}^{-1}]$-module spectra, and $\Phi^{C_2}(MU_{\mathbb{R}}[\overline{v_n}^{-1}]) \simeq *$.

Lemma 6.2.10. *Let $f : \mathcal{N} \to \mathcal{M}$ be an affine flat map of stacks and $\mathcal{F} \in \mathbf{QCoh}(\mathcal{N})$. If $\mathcal{M}$ admits a faithfully flat cover by an affine scheme (e.g. if $\mathcal{M}$ is the stack associated to a Hopf algebroid), there is an isomorphism*

$$H^*(\mathcal{M}; f_*\mathcal{F}) \cong H^*(\mathcal{N}; \mathcal{F})$$

Proof. The quasicoherent sheaf $\mathcal{F}$ admits an injective resolution in $\mathbf{QCoh}(\mathcal{N})$

$$\mathcal{F} \simeq (0 \to \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to \cdots)$$

If, for example, $\mathcal{N}$ is the stack associated to a Hopf algebroid, we may choose the $\mathcal{I}_i$'s to correspond to cofree comodules under the equivalence of Theorem 2.1.54. By definition (see 2.1.50), $H^*(\mathcal{N};\mathcal{F})$ is the cohomology of the complex

$$0 \to \Gamma(\mathcal{N};\mathcal{I}_0) \to \Gamma(\mathcal{N};\mathcal{I}_1) \to \Gamma(\mathcal{N};\mathcal{I}_2) \to \cdots$$

where $\Gamma(\mathcal{N};-)$ denotes the global sections functor. Since f is affine,

$$0 \to f_*\mathcal{I}_0 \to f_*\mathcal{I}_1 \to f_*\mathcal{I}_2 \to \cdots$$

is exact in $\mathbf{QCoh}(\mathcal{M})$, as f_* is exact; this may be checked on the affine cover of $\mathcal{M}$ given by assumption, where pushforward is given by restriction of scalars. Moreover, we have

$$\Gamma(\mathcal{M}; f_*\mathcal{I}_i) = \Gamma(\mathcal{N};\mathcal{I}_i)$$

It suffices to show then that each $f_*\mathcal{I}_i$ is injective, which follows from the adjunction

$$\mathrm{Hom}_{\mathbf{QCoh}(\mathcal{M})}(-, f_*\mathcal{I}_i) \cong \mathrm{Hom}_{\mathbf{QCoh}(\mathcal{N})}(f^*(-),\mathcal{I}_i)$$

and the fact that f^* is exact, as f is flat. □

Theorem 6.2.11. $\Phi(ER(n)) = 2^n$. *In particular,* $ER(n) \wedge X(2^n)$ *is complex orientable.*

Proof. We first prove that $\Phi(ER(n)) \leq 2^n$: let X be any homotopy commutative ring spectrum under $X(2^n)$ such that MU_*X is even (e.g. MU or $X(2^n)$ itself). Then we may form $\mathcal{M}_X$, and we have a commutative diagram

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{f}"'] & \mathcal{M}_X \arrow[d] \\
\mathcal{N} \arrow[r] \arrow[d, "f"'] & \mathcal{M}_{FG}(2^n) \arrow[d] \\
\mathrm{Spec}(E(n)_*)/C_2 \arrow[r, "p"] & \mathcal{M}_{FG}(1)
\end{tikzcd}$$

where both squares are pullbacks, $\mathcal{N}$ is an affine scheme, and f and p are affine morphisms by the proof of Proposition 6.2.7. By the commutativity of the diagram, we have

$$\begin{aligned} H^*(\mathrm{Spec}(E(n)_*)/C_2; p^*\mathcal{F}_X) &= H^*(\mathrm{Spec}(E(n)_*)/C_2; p^*(\phi_X)_*\mathcal{O}_{\mathcal{M}_X}) \\ &= H^*(\mathrm{Spec}(E(n)_*)/C_2; f_*(\tilde{f}_*\tilde{p}^*\mathcal{O}_{\mathcal{M}_X})) \\ &= H^0(\mathrm{Spec}(E(n)_*)/C_2; f_*(\tilde{f}_*\tilde{p}^*\mathcal{O}_{\mathcal{M}_X})) \\ &= H^0(C_2; (E(n) \wedge X)_*) \end{aligned}$$

where the higher cohomology groups vanish since one has an isomorphism

$$H^*(\mathrm{Spec}(E(n)_*)/C_2; f_*(\tilde{f}_*\tilde{p}^*\mathcal{O}_{\mathcal{M}_X})) \cong H^*(\mathcal{N}; \tilde{f}_*\tilde{p}^*\mathcal{O}_{\mathcal{M}_X})$$

by Lemma 6.2.10, as f is flat by Proposition 6.2.2. Now $\mathcal{N}$ is affine, and the higher cohomology of a quasicoherent sheaf on an affine scheme vanishes by Serre vanishing [85, tag 01XB].

By Lemma 6.2.9, the above sheaf cohomology is the E_2 page of the HFPSS computing $\pi_*((E_{\mathbb{R}}(n) \wedge i_*X)^{hC_2})$. The spectral sequence thus collapses and the result follows.

To see that $\Phi(ER(n)) \geq 2^n$, suppose to the contrary that $ER(n) \wedge X(2^n - 1)$ is complex orientable. We will see in Proposition 6.3.1 that $\mathcal{M}_{ER(n)} \simeq \mathrm{Spec}(E(n)_*)/C_2$, and so it follows that there is a pullback square

$$\begin{array}{ccc} \mathcal{M}_{ER(n)\wedge X(2^n-1)} & \longrightarrow & \mathcal{M}_{FG}(2^n - 1) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(E(n)_*)/C_2 & \longrightarrow & \mathcal{M}_{FG}(1) \end{array}$$

If the pullback $\mathcal{M}_{ER(n)\wedge X(2^n-1)}$ were affine, it would be discrete. This is a contradiction because we can consider the point of the pullback at the ring $\mathbb{F}_2$ given by

$$(E(n)_* \xrightarrow{\phi} \mathbb{F}_2, \phi^* F_{E(n)}, \mathrm{id})$$

where ϕ sends $v_i \mapsto 0$ for $i \neq n$ and $v_n \mapsto 1$. This has a nontrivial automorphism because

$$[-1]_{F_{E(n)}}(x) \equiv x \mod (2, v_1, \ldots, v_{n-1}, v_n - 1) + x^{2^n}$$

(see [11, Proposition 3.5]) $\square$

Remark 6.2.12. This recovers Hopkins' result that $KO \wedge X(2)$ is complex orientable. The proof gives something slightly stronger than stated, namely that if X is any ring spectrum under $X(2^n)$ so that MU_*X is even and commutative, then

$$(ER(n) \wedge X)_* = H^0(C_2; (E(n) \wedge X)_*)$$

Most of the same arguments above work for various homotopy fixed point spectra of Morava E-theories. We fix a prime p and let E_n be the Morava E-theory associated to a height n formal group law $\mathbb{G}$ over a perfect field k of characteristic p. Let G be a finite subgroup of the corresponding Morava stabilizer group, and set $EO_n := E_n^{hG}$. Let ν be the usual valuation on $\mathrm{End}(\mathbb{G})$ normalized so that $\nu(p) = 1$; we define:

$$N(G) := n \cdot \max\{\nu(g-1) : e \neq g \in G\}$$

Theorem 6.2.13. $\Phi(EO_n) = p^{N(G)}$*. In particular,* $EO_n \wedge X(p^{N(G)})$ *is complex orientable.*

Proof. Consider the pullback square

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d] & \mathcal{M}_{FG}(p^{N(G)}) \arrow[d] \\
\mathrm{Spec}(W(k)[[u_1, \dots, u_{n-1}]])/G \arrow[r, "\phi"] & \mathcal{M}_{FG}
\end{tikzcd}$$

We claim $\mathcal{M}$ is an affine scheme. Let F denote a p-typical universal deformation of $\mathbb{G}$, then for each $g \in G$, we have an isomorphism

$$[g] : F \to g^*F$$

of formal group laws over $W(k)[[u_1, \dots, u_{n-1}]]$, by universal property. We claim that if there is a ring map

$$f : W(k)[[u_1, \dots, u_{n-1}]] \to R$$

and $e \neq g \in G$ such that $f \circ g = f$, then it is impossible for the automorphism

$$f^*[g] : f^*F \to f^*F$$

to have the property that $(f^*[g])(x) \equiv x \mod x^{p^{N(G)}+1}$. Since we have chosen a p-typical coordinate, we may write

$$[g](x) = x +_{g^*F} \sum{}^{g^*F} t_i(g) x^{p^i}$$

Letting $\mathfrak{m}$ denote the maximal ideal in $W(k)[[u_1, \ldots, u_{n-1}]]$, for each $e \neq g \in G$, the inequality

$$n \cdot \nu(g-1) \leq p^{N(G)}$$

implies that $t_i(g)$ is a unit mod $\mathfrak{m}$ for some $i \leq N(G)$, hence $t_i(g)$ is also a unit in the local ring $W(k)[[u_1, \ldots, u_{n-1}]]$. But then if $(f^*[g])(x) \equiv x \mod x^{p^i+1}$, it follows that $f(t_i(g)) = 0$, a contradiction for a nonzero ring R.

The morphism ϕ is therefore affine, and hence by [65, Main Theorem], the even periodic-refinement of $\mathrm{Spec}(W(k)[[u_1, \ldots, u_{n-1}]])/G$ with global sections EO_n is what Mathew-Meier call a 0-affine derived stack, and in particular that $EO_n \wedge X \simeq (E_n \wedge X)^{hG}$ for any spectrum X [65, Proposition 4.11]. The proof now follows that of Theorem 6.2.11. □

Corollary 6.2.14. *Let $n = k(p-1)$ so that we have a tower $\mathbb{Q}_p \subset \mathbb{Q}_p(\zeta_p) \subset \mathrm{End}(\mathbb{G})[1/p]$. Then for all $0 < k < p$, $\zeta^k - 1$ is a uniformizer of $\mathcal{O}_{\mathbb{Q}_p(\zeta_p)}$, and since $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p$ is totally ramified, we therefore have $\nu(\zeta^k - 1) = \frac{1}{p-1}$, so that $N(C_p) = k$, and we find that*

$$\Phi(EO_{k(p-1)}) = p^k$$

6.3 Modular descriptions of $\mathcal{M}_{ER(n)}$

In this section, we study the stack $\mathcal{M}_{ER(n)}$ from a modular point of view. We provide the promised equivalence

$$\mathcal{M}_{ER(n)} \simeq \mathrm{Spec}(E(n)_*)/C_2$$

and use this to derive several other descriptions. At $n = 1$, we recover Hopkins nonsingular quadratic equations stack for $\mathcal{M}_{KO}$, and at $n = 2$, we relate this stack to elliptic curves with level structures, recovering results that are implicit in the work of [62] and [40].

Proposition 6.3.1. *Let $E \in \mathbf{Sp}^G$ be a G-ring spectrum with G-action so that $\pi_*^e(E)$ is even, and suppose the map p in the diagram*

$$\begin{tikzcd} \mathrm{Spec}(E_*) \arrow[r] \arrow[d] & \mathcal{M}_{FG}(1) \\ \mathrm{Spec}(E_*)/G \arrow[ur, "p"'] & \end{tikzcd}$$

is affine. Then if $MU^{\wedge s} \wedge E^{hG} \simeq (MU^{\wedge s} \wedge E)^{hG}$ for $s = 1, 2$, there is an equivalence of stacks $\mathcal{M}_{E^{hG}} \simeq \mathrm{Spec}(E_*)/G$.

Proof. Set $\mathcal{N} := \mathrm{Spec}(E_*)/G$. By affineness, we have a diagram of pullback squares

$$\begin{tikzcd} \mathrm{Spec}\Big(\Gamma(\mathcal{N} \times_{\mathcal{M}_{FG}(1)} \mathrm{Spec}(L) \times_{\mathcal{M}_{FG}(1)} \mathrm{Spec}(L); \mathcal{O})\Big) \arrow[r] \arrow[d] & \mathrm{Spec}\Big(\Gamma(\mathcal{N} \times_{\mathcal{M}_{FG}(1)} \mathrm{Spec}(L); \mathcal{O})\Big) \arrow[d] \\ \mathrm{Spec}\Big(\Gamma(\mathcal{N} \times_{\mathcal{M}_{FG}(1)} \mathrm{Spec}(L); \mathcal{O})\Big) \arrow[r] \arrow[d] & \mathcal{N} \arrow[d, "p"] \\ \mathrm{Spec}(L) \arrow[r] & \mathcal{M}_{FG}(1) \end{tikzcd}$$

and

$$\Gamma(\mathcal{N} \times_{\mathcal{M}_{FG}(1)} \mathrm{Spec}(L); \mathcal{O}) \cong \Gamma(\mathcal{N}; p^* \mathcal{F}_{MU}) \cong H^0(G; MU_* E) = MU_* E^{hG}$$

That the higher cohomology groups vanish follows from the same proof as used in the proof of 6.2.11. Therefore the top right stack is $\mathcal{M}_{MU \wedge E^{hG}}$, and the top left stack is $\mathcal{M}_{MU \wedge MU \wedge E^{hG}}$. By Theorem 2.1.44 we therefore have

$$\mathcal{N} \simeq \mathcal{M}_{((MU \wedge E^{hG})_*, (MU \wedge MU \wedge E^{hG})_*)} = \mathcal{M}_{E^{hG}}$$

All equivalences we passed through are $\mathbb{G}_m$-equivariant, so this equivalence is as $\mathbb{G}_m$-objects over $\mathcal{M}_{FG}(1)$. □

Corollary 6.3.2. *We have equivalences*

1. $\mathcal{M}_{ER(n)} \simeq \mathrm{Spec}(E(n)_*)/C_2$

2. $\mathcal{M}_{EO_n}/\mathbb{G}_m \simeq \mathrm{Spec}(W(k)[[u_1,\dots,u_{n-1}]])/G$

We focus now on $\mathcal{M}_{ER(n)}$ and give several explicit descriptions of it. For the rest of the section, we will implicitly work 2-locally everywhere.

Proposition 6.3.3. *As a $\mathbb{G}_m$-stack, $\mathcal{M}_{ER}(n)$ is equivalent to the stack associated to the graded Hopf algebroid*

$$(\mathbb{Z}[v_1,\dots,v_n^{\pm}],\mathbb{Z}[v_1,\dots,v_n^{\pm},r]/(r^2+v_nr))$$

where $|v_i| = 2(2^i-1)$ and $|r| = 2(2^n-1)$. Setting $r_i := \frac{v_i}{v_n}r$, we have the following formulae

$$\eta_R(v_i) = v_i + 2r_i$$

$$\epsilon(v_i) = v_i \qquad \epsilon(r) = 0$$

$$c(v_i) = v_i + 2r_i \qquad c(r) = -r$$

$$\Delta(r) = r\otimes 1 + 1\otimes r \qquad \Delta(v_i) = v_i\otimes 1$$

Proof. We have already shown that $\mathcal{M}_{ER(n)} \simeq \mathrm{Spec}(E(n)_*)/C_2$, so

$$\mathcal{M}_{ER(n)} \simeq \mathcal{M}_{(E(n)_*,\prod\limits_{g\in C_2} E(n)_*)}$$

as in Example 2.1.41. The above formulae are then read off from this Hopf algebroid after passing thru the isomorphism

$$\mathbb{Z}[v_1,\dots,v_n^{\pm},r]/(r^2+v_nr) \to \prod_{g\in C_2} E(n)_*$$

that sends $r\mapsto(0,-v_n)$. The formula for $\eta_R(v_n)$ is clear, and $\eta_R(v_i)$ follows from the fact that

$$\eta_R(v_i/v_n) = \eta_L(v_i/v_n)$$

since $\gamma(v_i/v_n) = v_i/v_n$. $\square$

By setting $a_{2^i-1} := v_i$, we obtain the following modular description of $\mathcal{M}_{ER(n)}$.

Proposition 6.3.4. $\mathcal{M}_{ER(n)}$ *is the stackification of the functor which assigns to a ring R, the groupoid of n-tuples of quadratic equations*

$$(q_1(x), \ldots, q_n(x)) = (x^2 + a_1 x, \ldots, x^2 + a_{2^n-1}x)$$

over R, so that $a_{2^n-1} \in R^\times$, with a morphism

$$(x^2 + a_1 x, \ldots, x^2 + a_{2^n-1}x) \to (x^2 + b_1 x, \ldots, x^2 + b_{2^n-1}x)$$

consisting of $r \in R$ such that $q_n(r) = 0$, and $b_{2^i-1} = a_{2^i-1} + 2r_{2^i-1}$ where $r_{2^i-1} = \frac{a_{2^i-1}}{a_{2^n-1}} r$. In other words, a morphism with domain $(q_1, \ldots, q_n)$ is simply a choice of coordinate transformation $x \mapsto x + r$ that preserves the condition $q_n(0) = 0$.

Proof. This description is immediate from Proposition 6.3.3 once one observes that if $q_n(r) = 0$, $q_i(r_{2^i-1}) = 0$ automatically. Indeed

$$a_{2^n-1}(r_{2^i-1}^2 + a_{2^i-1} r_{2^i-1}) = a_{2^n-1}\left(\frac{a_{2^i-1}^2}{a_{2^n-1}}\left(\frac{r^2}{a_{2^n-1}} + r\right)\right) = \frac{a_{2^i-1}^2}{a_{2^n-1}}\left(r^2 + a_{2^n-1} r\right) = 0$$

and a_{2^n-1} is a unit. $\square$

Remark 6.3.5. This recovers Hopkins' description of $\mathcal{M}_{KO}$ when $n = 1$ (see [22]).

At $n = 1$, Atiyah's $(K_{\mathbb{R}})_{(2)}$ is a form of $E_{\mathbb{R}}(1)$, and at $n = 2$, it is a result of Hill-Meier that the C_2-spectrum $Tmf_1(3)[\overline{a_3}^{-1}]$ is a form of $E_{\mathbb{R}}(2)$ [40]. In the latter case, this spectrum is defined as the global sections of a derived stack, and so it is already related to a particular moduli problem. More specifically,

$$Tmf_1(3)[\overline{a_3}^{-1}] = \mathcal{O}^{TMF}(\overline{\mathcal{M}_1(3)}[a_3^{-1}])$$

where $\mathcal{O}^{TMF}$ is the TMF sheaf. We have a $\mathbb{G}_m$-equivariant diagram of pullback squares as follows:

$$\begin{array}{ccccc}
\mathcal{M}_1^1(3) & \longrightarrow & \mathrm{Spec}(Tmf_1(3)[a_3^{-1}]_*) & \longrightarrow & \overline{\mathcal{M}_1^1(3)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_0^1(3) & \longrightarrow & \overline{\mathcal{M}_0^1(3)}[a_3^{-1}] & \longrightarrow & \overline{\mathcal{M}_0^1(3)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}_{ell}^1 & \longrightarrow & \overline{\mathcal{M}_{ell}^1}[a_3^{-1}] & \longrightarrow & \overline{\mathcal{M}_{ell}^1}
\end{array}$$

The superscript 1's denote that we are working with the corresponding $\mathbb{G}_m$-torsors over each particular moduli problem, e.g. $\mathcal{M}_0^1(3)$ here is the moduli stack of elliptic curves with $\Gamma_0(3)$-structure and a chosen nonvanishing 1-form, or equivalently with *strict* coordinate changes. The top row of vertical maps are all C_2-torsors, and all stacks in the top row are actually schemes:

$$\mathcal{M}_1^1(3) = \mathrm{Spec}(\mathbb{Z}[a_1, a_3][\Delta^{-1}])$$

$$\mathrm{Spec}(Tmf_1(3)[a_3^{-1}]_*) = \mathrm{Spec}(\mathbb{Z}[a_1, a_3^{\pm}]) = \overline{\mathcal{M}_1^1(3)}[a_3^{-1}]$$

$$\overline{\mathcal{M}_1^1(3)} = \mathrm{Spec}(\mathbb{Z}[a_1, a_3]) \smallsetminus (V(a_1) \cap V(a_3))$$

where $\Delta = a_3^3(a_1^3 - 27a_3)$. Since $Tmf_1(3)[\overline{a_3}^{-1}]$ is a form of $E_{\mathbb{R}}(2)$, it follows that the stack

$$\mathcal{M}_{(Tmf_1(3)[\overline{a_3}^{-1}])^{hC_2}}$$

is equivalent to both $\overline{\mathcal{M}_0^1(3)}[a_3^{-1}]$ and the 2-tuples of quadratic equations stack in Proposition 6.3.4, and we would like to describe this equivalence more explicitly.

Every elliptic curve C with a point P of exact order 3 admits a unique strict isomorphism to a curve E of the form $y^2 + a_1xy + a_3y = x^3$ sending P to $(0,0)$, which is the content of the equivalence $\mathcal{M}_1^1(3) \simeq \mathrm{Spec}(\mathbb{Z}[a_1, a_3][\Delta^{-1}])$ [62, Proposition 3.2]. The formal inversion $[-1]$ on E sends $(0,0)$ to $(0, -a_3)$, and so if we define $c(E)$ to be the curve $y^2 - a_1xy - a_3y = x^3$, the inverse of the unique strict isomorphism $c(E) \to E$ sending the torsion point $(0, a_3)$ to $(0,0)$ is of the form

$$x \mapsto x$$

$$y \mapsto y - a_1x - a_3$$

This shows that C_2 acts on $\mathcal{M}_1^1(3)$ by sending $a_1 \mapsto -a_1$ and $a_3 \mapsto -a_3$. These are the same formulae as those of $\mathcal{M}_{ER(2)}$, so we see that transformations of the form

$$(x^2 + a_1x, x^2 + a_3x) \xrightarrow{r} (x^2 + (a_1 + 2\frac{a_1r}{a_3})x, x^2 + (a_3 + 2r)x)$$

with $r^2 + a_3 r = 0$ can be repackaged as transformations

$$\{y^2 + a_1xy + a_3y = x^3\} \xrightarrow{r} \{y^2 + (a_1 + 2\frac{a_1 r}{a_3})xy + (a_3 + 2r)x = x^3\}$$

via $x \mapsto x$, $y \mapsto y + \frac{a_1 r}{a_3}x + r$. This leads us to a generalization for higher $ER(n)$.

Proposition 6.3.6. *There is a $\mathbb{G}_m$-equivariant equivalence of stacks*

$$\mathcal{M}_{ER(n)} \simeq \begin{cases} \bullet \text{ Curves } C = \{x_n^2 + a_1 x_{n-1}x_n + \cdots + a_{2^{n-1}-1}x_1x_n + a_{2^n-1}x_n = x_1^2 x_{n-1}\} \text{ with } a_{2^n-1} \in R^\times \\ \bullet \text{ Strict coordinate changes preserving the intersection with the line } \{(0,\dots,0,x_n)\} \end{cases}$$

where $|a_i| = 2i$.

Proof. Note that the equation defining C is a homogeneous equation of degree $4(2^n - 1)$ if we give $|x_{n-k}| = 2(2^n - 2^k)$ and $|a_k| = 2k$. We consider strict coordinate changes of the form

$$x_1 \mapsto x_1 + r_{11}$$

$$x_2 \mapsto x_2 + r_{21}x_1 + r_{22}$$

$$x_3 \mapsto x_3 + r_{31}x_2 + r_{32}x_1 + r_{33}$$

$$\vdots$$

$$x_n \mapsto x_n + r_{n1}x_{n-1} + \cdots + r_{n,n-1}x_1 + r_{nn}$$

However, the intersection of C with the line $\{(0,\dots,0,x_n)\}$ gives the equation $x_n^2 + a_{2^n-1}x_n = 0$, and so if we restrict to strict transformations as above that preserve this intersection, we find that the only possibilities are transformations of the form

$$x_1 \mapsto x_1$$

$$\vdots$$

$$x_{n-1} \mapsto x_{n-1}$$

$$x_n \mapsto x_n + \frac{ra_1}{a_{2^n-1}}x_{n-1} + \cdots + \frac{ra_{2^{n-1}-1}}{a_{2^n-1}}x_1 + r$$

with $r^2 + a_{2^n-1}r = 0$, and this is a strict isomorphism of curves from

$$C = \{x_n^2 + a_1 x_{n-1}x_n + a_3 x_{n-2}x_n + a_7 x_{n-3}x_n + \cdots + a_{2^{n-1}-1}x_1 x_n + a_{2^n-1}x_n = x_1^2 x_{n-1}\}$$

to the curve

$${}^rC = \{x_n^2 + (a_1 + 2\frac{ra_1}{a_{2^n-1}})x_{n-1}x_n + \cdots + (a_{2^{n-1}-1} + 2\frac{ra_{2^{n-1}-1}}{a_{2^n-1}})x_1x_n + (a_{2^n-1} + 2r)x_n = x_1^2 x_{n-1}\}$$

□

Remark 6.3.7. When $n = 1$, we set $x_{n-1} = 0$ by convention and recover the nonsingular quadratic equations stack. Instead of asking our transformations to preserve the intersection with the line $\{(0,\ldots,0,x_n)\}$, we could have equivalently asked it to preserve the set of points $\{(0,\ldots,0),(0,\ldots,-a_{2^n-1})\}$ - these sheafify to give the same thing. When $n = 2$, $\{(0,0),(0,-a_{2^n-1})\}$ is the subgroup generated by the 3-torsion point $\{(0,0)\}$ on $y^2+a_1xy+a_3 = x^3$, so this stack is equivalent to $\overline{\mathcal{M}_0^1(3)}[a_3^{-1}]$.

Proposition 6.3.6 expresses the stack $\mathcal{M}_{ER(n)}$ in terms of some sort of arithmetic data, and so there should be some sort of modular description of the map $\mathcal{M}_{ER(n)} \to \mathcal{M}_{FG}(1)$ in terms of this data. At $n = 1$, quadratic equations are encoding flat C_2-torsors, and such a description is given in [64, Section 3.7]. At $n = 2$, we have the usual description of sending an elliptic curve to its associated formal group, and the inversion map on the elliptic curve corresponds to the formal inversion $[-1]$ on its associated formal group. We have a third description of this stack where such a thing has been studied at higher heights.

Proposition 6.3.8. *There is a $\mathbb{G}_m$-equivalence of stacks:*

$$\mathcal{M}_{ER(n)} \simeq \begin{cases} \bullet\ \textit{Curves } C = \{y^{2^n-1} = x^2 + \sum_{i=1}^{n} a_{2^i-1}xy^{2^{n-1}-2^{i-1}}\} \textit{ with } a_{2^n-1} \in R^\times \\ \bullet\ \textit{Strict coordinate changes } y \mapsto y, x \mapsto x + \sum_{i=1}^{n} t_i y^{2^{n-1}-2^{i-1}} \\ \textit{preserving the points } \{(0,0),(-a_{2^n-1},0)\} \end{cases}$$

where $|a_i| = 2i$.

Proof. The proof is as before, we are simply reparametrizing the quotient stack

$$\mathrm{Spec}(E(n)_*)/C_2$$

in several different ways. The v_i's again correspond to the coefficients a_{2^i-1}, and one checks that any strict coordinate change preserving the points $\{(0,0),(-a_{2^n-1},0)\}$ has the form

$$t_i = \frac{r a_i}{a_{2^n-1}}$$

for r satisfying $r^2 + a^{2^n-1} r = 0$. □

Remark 6.3.9. The curves in Proposition 6.3.8 are special cases of curves that have been studied by Ravenel, Gorbunov and Mahowald [28] [81]. For example at height 3, we have $y^7 = x^2 + a_1 xy^3 + a_3 xy^2 + a_7 x$, and they study more generally curves of the form

$$y^7 = x^2 + a_1 xy^3 + a_3 xy^2 + a_7 x + b_1 y^6 + b_2 y^5 + b_3 y^4 + b_4 y^3 + b_5 y^2 + b_6$$

and show that their associated Jacobians admit a 1-dimensional summand of their formal groups, and the resulting formal group is Landweber exact over the ring $\mathbb{Z}_2[a_i, b_j]$.

6.4 Future directions with chromatic measure and $\mathcal{M}_{E^G}$

In the previous section, we recovered Hopkins' description of $\mathcal{M}_{ER(1)}$ as the moduli stack of nonsingular quadratic equations. However, he also gives a description of the stack associated to the connective cover $\mathcal{M}_{ko}$ as the moduli stack of *all* quadratic equations. It is a goal of the author's in future work to establish analogous results for the stacks $\mathcal{M}_{BP_{\mathbb{R}}\langle n\rangle^{C_2}}$. A significant step in this direction would be a computation of $\Phi(BP_{\mathbb{R}}\langle n\rangle^{C_2})$. We have substantial evidence of the following.

Conjecture 6.4.1. $\Phi(BP_{\mathbb{R}}\langle n\rangle^{C_2}) = 2^n$. *In particular,* $BP_{\mathbb{R}}\langle n\rangle^{C_2} \wedge X(2^n)$ *is complex orientable.*

Our approach is to use the slice tower of $BP_{\mathbb{R}}\langle n\rangle$. Since we are working 2-locally, it suffices to use the spectrum $T(n)$ in place of $X(2^n)$ (see [77, Section 6.5]). The slice associated graded of $BP_{\mathbb{R}}\langle n\rangle$ is

$$H\underline{\mathbb{Z}}_{(2)}[\overline{v_1},\ldots,\overline{v_n}]$$

so by smashing the slice tower with the C_2-spectrum $i_*T(n)$, one has a spectral sequence

$$\pi_\bigstar(i_*T(n)\wedge\underline{\mathbb{Z}}_{(2)})[\overline{v_1},\ldots,\overline{v_n}] \implies \pi_\bigstar(i_*T(n)\wedge BP_{\mathbb{R}}\langle n\rangle)$$

Note in particular that $(i_*T(n)\wedge BP_{\mathbb{R}}\langle n\rangle)^{C_2}\simeq T(n)\wedge(BP_{\mathbb{R}}\langle n\rangle)^{C_2}$, so it suffices to show that in the integer-graded part of this spectral sequence, the E_∞ page is concentrated in even stems. Comparing this spectral sequence with its localized variant, the author has shown a pattern of differentials for small values of n such that every odd dimensional class supports or receives a differential. In joint work in progress with Mike Hill and Doug Ravenel, the author has made significant progress toward a calculation of $H_*(BP_{\mathbb{R}}\langle n\rangle^{C_2};\mathbb{F}_2)$ as an $\mathcal{A}_*$-comodule. These computations also suggest that this conjecture may be true.

Similar questions may be asked of course for the $BP^{((G))}\langle m\rangle$'s as in [11]. Computations with these theories become correspondingly difficult as $|G|$ grows, but these analyses in the base case $G=C_2$ are important steps toward understanding the theories with larger groups of equivariance.

Remark 6.4.2. The definition of the chromatic measure integer $\Phi(E)$ was inspired in part by a similar integer defined by Chatham and Bhattacharya in [19] for E a ring spectrum:

$$\Theta(E)=\min\{n>0:\ \gamma_1^{\oplus n}\text{ is }E\text{-orientable}\}$$

where γ_1 is the universal complex line bundle over $\mathbb{CP}^\infty$. As far as the author is aware, the integers $\Phi(E)$ and $\Theta(E)$ coincide in every case in which both is known. $\Phi(E)$ is much easier to compute than $\Theta(E)$ with an understanding of $\mathcal{M}_E$ as a moduli problem. Chatham and Bhattacharya relate Θ to A_∞-$MU[n]$ orientations where

$$MU[n]=\mathrm{Thom}(BU\xrightarrow{n}BU)$$

Some analysis of $\mathcal{M}_{MU[n]}$ has suggested a possible relationship between these two numbers, but the author is not aware of any direct connection.

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