

Stacks & Chromatic Homotopy Theory

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The goal of this seminar is to study the chromatic picture of stable homotopy theory via the language of stacks. The basic idea here is that complex bordism provides a functor from the stable homotopy category to the category of quasicoherent sheaves on \mathcal{M}_{FG} , the moduli stack of formal groups, and this functor turns out to be a surprisingly strong approximation.

We will begin with generalities on stacks, and then spend most of the seminar in a deep study of the geometry of \mathcal{M}_{FG} . This portion of the seminar will involve almost no topology / homotopy theory, and although we focus mostly on this one stack, many of the techniques we will discuss are very general. The topics here also run parallel to the number theory seminar this quarter. We will wrap up the seminar by discussing how to pull back features of \mathcal{M}_{FG} to the stable homotopy category.

Below are brief summaries of each talks, but I will send each speaker in advance a more detailed set of notes on what to cover and how to navigate the literature.

Talk 1. *Stacks*

Grothendieck topologies and sheaves on a Grothendieck site. Sheafification via the plus construction. Define a stack. Stackification. Give detailed BG example, and explain quotient stacks. 2-category pull-back and stacky pullback. State general principle that when we say a morphism of stacks $f : \mathcal{M} \rightarrow \mathcal{N}$ has algebraic geometry property P , we mean that it is representable and every morphism $\mathrm{Spec}(R) \rightarrow \mathcal{N}$ pulls back along f to a morphism of schemes with property P . Give the example of a G -torsor and show that $\mathrm{Spec}(\mathbb{Z}) \rightarrow BG$ is a G -torsor.

Sources: [COCTALOS] lectures 8-10, The Stacks Project

Talk 2. *Locally Presentable Stacks*

(co)-Groupoid objects / Hopf algebroids. Explain how to get a Hopf Algebroid (and therefore a stack) from an Adams flat ring spectrum. Recast BG as a stack associated to a Hopf Algebroid. Every R -point of $\mathcal{M}_{(A,\Gamma)}$ flat locally factors thru the defining cover $\mathrm{Spec}(A) \rightarrow \mathcal{M}_{(A,\Gamma)}$. Locally presentable stacks admitting a flat cover are the same thing as stacks associated to a Hopf Algebroid.

Sources: [COCTALOS] lectures 9-10

Talk 3. \mathcal{M}_{FG}

Define a formal group law and (strict) isomorphisms thereof. Define the Lazard ring and the two Hopf Algebroids that give rise to $\mathcal{M}_{FG}^{(1)}$ and \mathcal{M}_{FG} . Discuss the difference between R -points of \mathcal{M}_{FG} and of $\mathcal{M}_{FG}^{(1)}$: the latter does not need stackification. The former does - explain 2 ways of looking at an R -point of \mathcal{M}_{FG} : the definition in Lurie as a group structure on nilpotents, and the COCTALOS $R[[P]]$ definition. Explain that the obstruction to \mathcal{M}_{FG}^{pre} being a stack is precisely the nontriviality of the line bundle ω . Wrap this all up by showing that $\mathcal{M}_{FG}^{(1)} \rightarrow \mathcal{M}_{FG}$ is a \mathbb{G}_m -torsor classified by $\omega : \mathcal{M}_{FG} \rightarrow B\mathbb{G}_m$. If time, describe the coordinate tower $\{\mathcal{M}_{FG}^{(n)}\}$ under $\mathrm{Spec}(L)$.

Sources: [Lurie] lecture 10, [COCTALOS] lecture 15

Talk 4. Lazard's Theorem

Prove that the Lazard ring is a polynomial ring. Follow the proof in Lurie lectures 2-3 and/or COCTALOS lectures 2-3.

Sources: [Lurie] lectures 2-3, [COCTALOS] lectures 2-3

Talk 5. $(\mathcal{M}_{FG})_{(p)}$

Define the height of a formal group law, the v_n 's, and show $\text{Spec}(\mathbb{Z}_{(p)}[v_1, v_2, \dots]) \rightarrow (\mathcal{M}_{FG})_{(p)}$ is a flat cover. What does height look like upon stackification - i.e. what is the height of a formal group - follow Goerss. Define the height filtration of $(\mathcal{M}_{FG})_{(p)}$. Show that $\text{Spec}(\mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm}]) \rightarrow \mathcal{M}_{FG}^{\leq n}$ is a flat cover. State Lazard's theorem on classification by height over algebraically closed fields. Define the algebraic space associated to a stack - show that the algebraic space associated to $(\mathcal{M}_{FG})_{(p)}$ has a single point for each height, and the topology of the space is given by the height filtration. We see for example that the height filtration is exhaustive.

Sources: [Lurie] lectures 12-13, 15, [COCTALOS] lecture 13, [Goerss] section 5

Talk 6. Morava Stabilizer Group

We now have a clear picture of the algebraic space associated to $(\mathcal{M}_{FG})_{(p)}$ - if we wanted to recover $(\mathcal{M}_{FG})_{(p)}$ from this space, we would need to know the automorphism group of each point of this space - these are the Morava stabilizer groups. Follow Lurie Lecture 19 and COCTALOS lecture 14.

Sources: [Lurie] lecture 19, [COCTALOS] lecture 14, [Ravenel] section 4.2

Talk 7. Lubin-Tate theory

We now know the automorphism groups of the points of our stack $(\mathcal{M}_{FG})_{(p)}$. Formal neighborhoods of points in a stack are given by deformations of the moduli object in question. Lubin-Tate theory tells us that, for formal groups, the deformation problem is unobstructed. In particular, the deformation spaces in question are discrete. Follow Lurie lecture 21.

Sources: [Lurie] lecture 21, [COCTALOS] lecture 13

Talk 8. Landweber Exact Functor Theorem

*Define a quasicoherent sheaf on a stack. State without proof the equivalence with comodules when $\mathcal{M} = \mathcal{M}_{(A, \Gamma)}$. Prove the stacks version of the LEFT along the lines of Lurie lecture 15. Then prove the algebraic version (with the v_i 's) following COCTALOS lecture 21. Invoke LEFT to build BP , $E(n)$, $\widehat{E(n)}$, and now we can easily define $K(n)$. Show that if E and F are Landweber exact, then E_*F is the ring of functions on the stacky pullback, as in Lurie lecture 18.*

Sources: [Lurie] lectures 15, 16, and 18, [COCTALOS] lecture 21

Talk 9. Bousfield Localization and \mathcal{M}_{FG}

Explain the basic setup of Bousfield localization in Spectra. Prove Theorem 3.3 in [Bauer]. Show how we can use the results of lecture 8 to import our robust picture of $(\mathcal{M}_{FG})_{(p)}$ into homotopy theory - this is our big payoff (for the homotopy theorists anyway). Construct the chromatic tower and prove the chromatic fracture squares are homotopy pullbacks. State the convergence of the chromatic tower. The "chromatic approximation functor" is a complete invariant on the compact objects in $Sp_{(p)}$. For general ring spectra it detects all nilpotence.

Sources: [Bauer], [Lurie] lectures 20, 23, 25, 26, 32

References

[Bauer] *Bousfield Localization and the Hasse Square*, Tilman Bauer

[COCTALOS] *Complex Oriented Cohomology Theories and the Language of Stacks*, Mike Hopkins

[Goerss] *Quasicoherent Sheaves on the Moduli Stack of Formal Groups*, Paul Goerss

[Lurie] *Lectures on Chromatic Homotopy Theory Math 252x*, Jacob Lurie

[Ravenel] *Nilpotence and Periodicity in Stable Homotopy Theory*, Doug Ravenel