Techniques in filtered and synthetic homotopy theory

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February 2, 2026

(joint work with Jack Davies and Sven van Nigtevecht)

Overview

- Sven's talk: tmf, the circularity in the literature, filtered homotopy groups and the τ-formalism.

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- Jack's talk: the page-by-page computation of the descent spectral sequence of Tmf.

- My talk: survey the tools that filtered/synthetic homotopy theory gives you to help compute a spectral sequence.

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$$\cdots \xrightarrow{\tau} X^1 \xrightarrow{\tau} X^0 \xrightarrow{\tau} X^{-1} \xrightarrow{\tau} \cdots$$

$$\mathrm{gr}^0 X$$

with ∂ mapping from $\mathrm{gr}^0 X$ back to X^1 and a vertical map from X^0 to $\mathrm{gr}^0 X$.

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- $\pi_{n,s}X = \pi_n X^{n+s}$ and $|\tau| = (0, -1)$

Tool 1: The τ-formalism and bigraded homotopy groups

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$$\text{Associated graded} \qquad \text{Filtration} \qquad \text{Abutment}$$

$$X/\tau \xleftarrow{\qquad \pi \qquad} X \xrightarrow{\qquad\qquad} \tau^{-1}X$$

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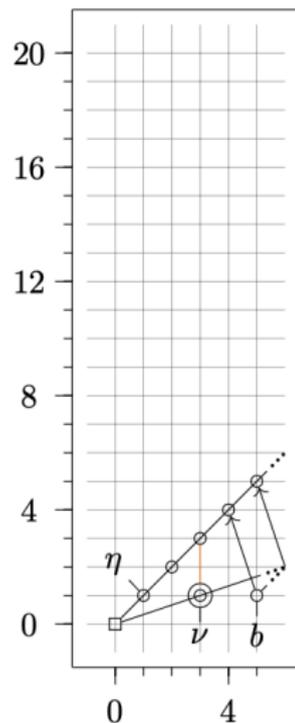- Extensions are encoded in τ relations, e.g.

$$xy = \tau^r z$$
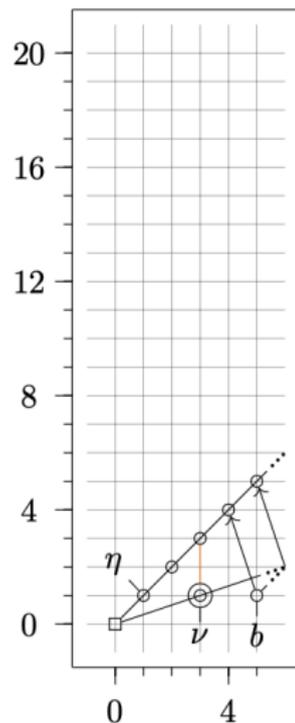
with $r > 0$ and $\pi(x), \pi(y), \pi(z) \neq 0$.

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Example: The Adams–Novikov spectral sequence

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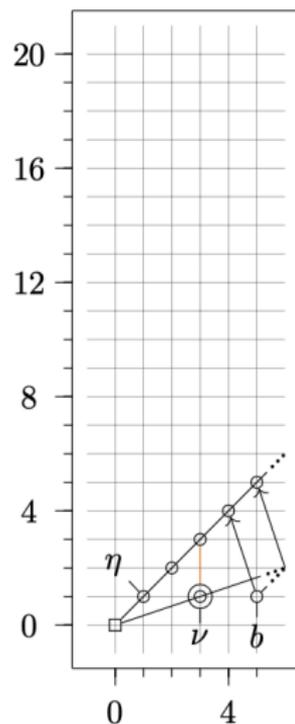- $\pi_{1,*} = \mathbb{F}_2[\tau]\{\eta\}$

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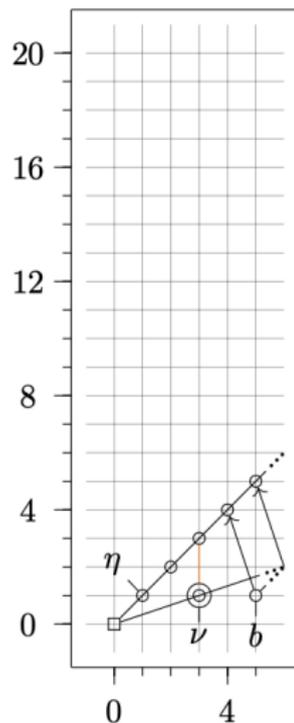- $\pi_{2,*} = \mathbb{F}_2[\tau]\{\eta^2\}$

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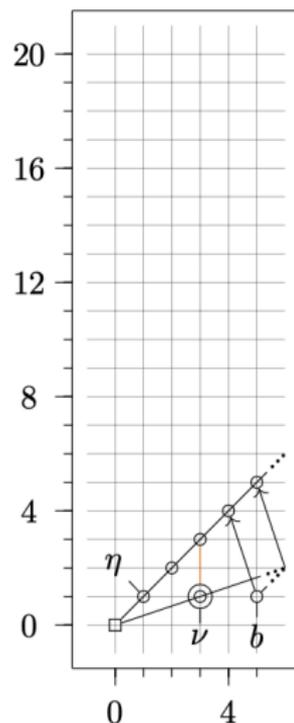- $\pi_{2,*} = \mathbb{F}_2[\tau]\{\eta^2\}$
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- $\pi_{4,*} = \mathbb{F}_2[\tau]\{\eta^4\}/(\tau^2\eta^4)$

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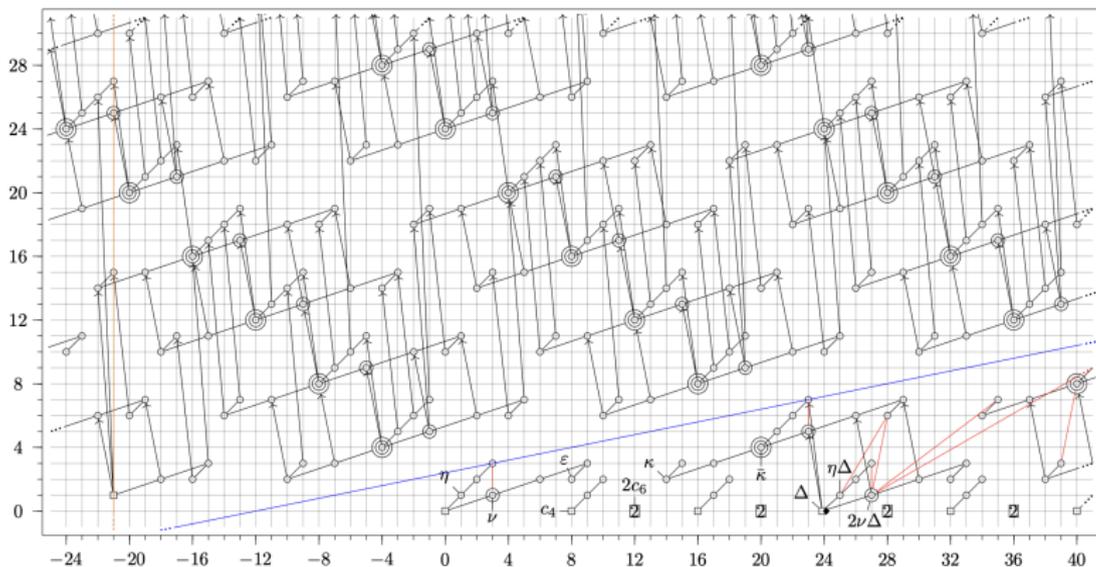- This is not clear in $\mathrm{DSS}(Tmf)$.

Figure: E_5-page of $\mathrm{DSS}(Tmf)$

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- Upshot: we can build good spectral sequences by taking limits and colimits.
- Modding out by τ commutes with all limits and colimits.
- Inverting τ commutes with all colimits.

- Want to compute

$$\cdots \to \pi_* \mathbb{S}^2 \xrightarrow{\eta^2} \pi_* \mathbb{S} \to \pi_* (\mathbb{S}/\eta^2) \to \pi_{*-1} \mathbb{S}^2 \to \cdots$$

Example: Designer SS for $\mathbb{S}/\eta^2$

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- Better, take $\eta^2 \in \pi_{2,2}\mathrm{ANSS}(\mathbb{S})$ and form $\mathrm{ANSS}(\mathbb{S})/\eta^2$.

- There is a map $\mathrm{ANSS}(\mathbb{S}) \to \mathrm{ANSS}(\mathit{Tmf})$, but $E_2 - \mathrm{ANSS}(\mathit{Tmf})$ is bad.

Example: Synthetic modular forms

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- $Tmf = \varprojlim_i A_i$, for A_i elliptic cohomology theories.
- We have natural maps of $\mathbb{E}_\infty$-algebras $\mathrm{ANSS}(\mathbb{S}) \to \mathrm{ANSS}(A_i)$.
- This defines a map of $\mathbb{E}_\infty$-algebras $\mathrm{ANSS}(\mathbb{S}) \to \varprojlim_i \mathrm{ANSS}(A_i) = Smf$, and Smf implements $\mathrm{DSS}(Tmf)$ (CDvN).

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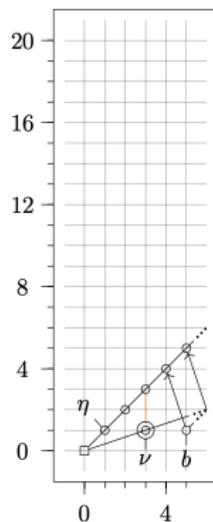- Upshot: $\tau^2\eta^3 = 4\nu$ and $\tau^2\eta^4 = 0$ on the nose in $\pi_{*,*}Smf$.

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Figure: ANSS of $\mathbb{S}_{(2)}$ through degree 4

Figure: DSS(Tmf)

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Tool 3: Detection methods

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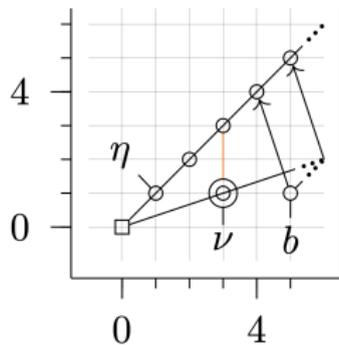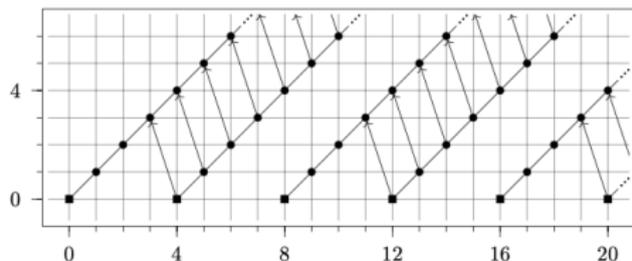- $2\eta = 0$, so $\eta = \partial(v_1)$

$$\cdots \to \pi_{*+1}\mathbb{S} \to \pi_{*+1}(\mathbb{S}/2) \xrightarrow{\partial} \pi_*\mathbb{S} \xrightarrow{2} \pi_*\mathbb{S} \to \cdots$$

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Example: detecting η in $\mathrm{ANSS}(ko)$.

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v_1 \downarrow & \mathrm{ANSS}(\mathbb{S})/(2,\tau) & \longrightarrow & \mathrm{ANSS}(ko)/(2,\tau) \;\; \neq 0 & \longrightarrow & \mathrm{ANSS}(ku)/(2,\tau) \;\; \beta \neq 0 \\
& \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
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Tool 3: Detection methods

- We didn't need a cobar complex for *ko*!
- Using the $\mathbb{E}_\infty$-map $\mathrm{ANSS}(\mathbb{S}) \to Smf$, we can repeat this argument. Using that Smf is a module over $\mathrm{ANSS}(\mathbb{S})$, we can generalize this to Miller–Ravenel–Wilson higher Greek letter families.

$$\mathbb{S}/(2^{i_0}, v_1^{i_1}, \ldots, v_h^{i_h}) \xrightarrow{\partial} \cdots \to \mathbb{S}/2^{i_0} \xrightarrow{\partial} \mathbb{S}$$

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The boundary map in the cofiber sequence

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is called the *total differential* of X.

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δ_1^∞

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- $d_5(\Delta) = \nu\bar{\kappa}$ in $\mathrm{DSS}(Tmf)$

Figure: $\mathrm{DSS}(Tmf)$
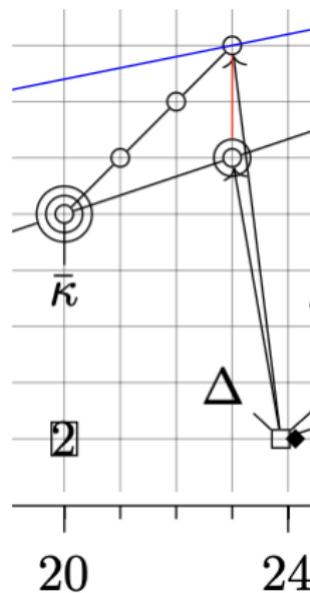
- $d_5(\Delta) = \nu\bar{\kappa}$ in $\mathrm{DSS}(\mathit{Tmf})$
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- $d_5(\Delta) = \nu\bar{\kappa}$ in $\mathrm{DSS}(Tmf)$
- $\delta_1^\infty(\Delta) = \nu\bar{\kappa}\tau^3$
- $4\nu = \tau^2\eta^3 \implies$

$$\delta_1^\infty(4\Delta) = 4\nu\bar{\kappa}\tau^3 = \eta^3\bar{\kappa}\tau^5$$

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- $d_7(4\Delta) = \eta^3\bar{\kappa}.$

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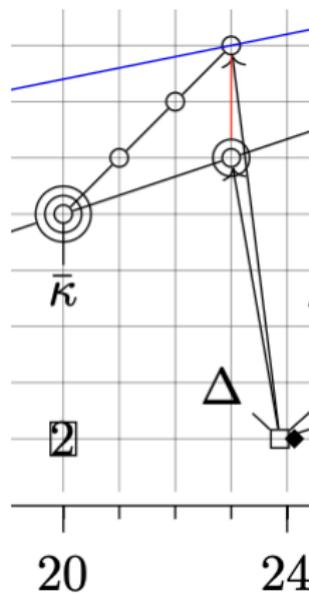
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- Solution: artificially force these to hold by replacing $\pi_{*,*}X$ with $\pi_{*,*}(X/\tau^n)$.

Tool 5: Working mod τ^n

- Slogan: the filtered spectrum X/τ^n terminates the spectral sequence on the E_{n+1}-page in a homotopy coherent way.
- Problem: we would like to use information from $E_2, \ldots, E_6$ to deduce a d_7. But the spectral sequence only gives us information mod higher filtration.
- So we have to go to the end of the spectral sequence to know:
  - If a class x is a permanent cycle, i.e. if it admits a lift to $\pi_{*,*}X$ along π.
  - If a relation like $\nu\bar{\kappa}\tau^4 = 0$ ($\iff \delta_1^\infty(\Delta) = \nu\bar{\kappa}\tau^3$) holds on the nose.
  - If a hidden extension $xy = \tau^r z$ holds on the nose.
- Solution: artificially force these to hold by replacing $\pi_{*,*}X$ with $\pi_{*,*}(X/\tau^n)$.
- Purely filtered technique: $\tau^{-1}(X/\tau^n) = 0$.

Tool 5: Working mod τ^n

Theorem (CDvN, Truncated omnibus theorem)

The bigraded homotopy groups $\pi_{*,*}(X/\tau^n)$ encode the data of the spectral sequence corresponding to X terminated on E_{n+1} with filtrations truncated to length $\leq n$. (i.e. we set $E_{n+1} = E_\infty$ and truncate filtrations).

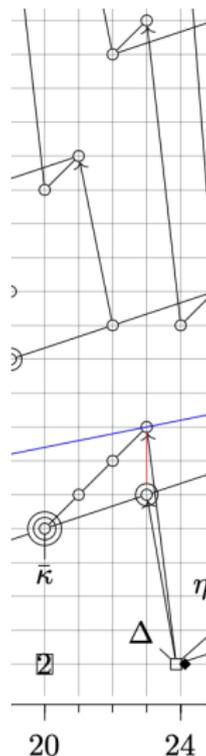## Theorem (CDvN, Truncated omnibus theorem)

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- More subtle than ordinary Omnibus (e.g. due to the Toda brackets $\langle x, \tau^r, \tau^{n-r} \rangle$ for $x\tau^r = 0$)

- We claimed that $\delta_1^\infty(\Delta) = \nu\bar\kappa\tau^3$.

Example: Truncated total differentials and $d_7(4\Delta)$

Definition

For $m \geq n$, the boundary map in the cofiber sequence

$$\Sigma^{0,-n} X/\tau^{m-n} \xrightarrow{\tau^n} X/\tau^m \to X/\tau^n \xrightarrow{\delta_n^m} \Sigma^{1,-n-1} X/\tau^{m-n}$$

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- $\delta_n^m(x) = \tau^r y$ for $r + n < m \implies d_{r+n+1}(x) = \pi(y)$

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- $d_7(4\Delta) = \eta^3\bar{\kappa}$.

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Tool 6: The generalized Leibniz rule

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The tower

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Theorem (Burklund)

The map $\mathbb{1}/\tau^{2n} \to \mathbb{1}/\tau^n$ is a square-zero extension of $\mathbb{E}_\infty$-rings.

- Upshot: For R a filtered ring spectrum, the map

$$\delta_n^{2n} : \pi_{*,*}(R/\tau^n) \to \pi_{*,*}(\Sigma^{1,-n-1}R/\tau^n)$$

is a $\pi_{*,*}(R/\tau^{2n})$-linear derivation.
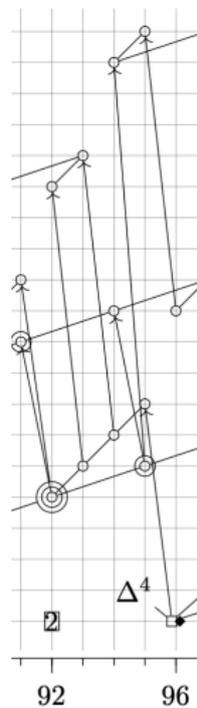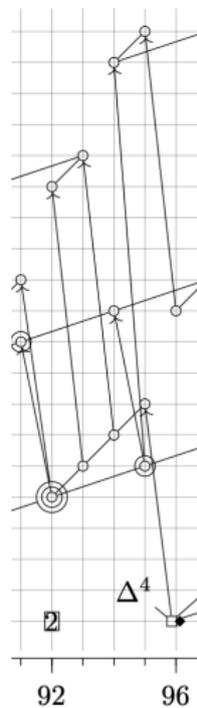
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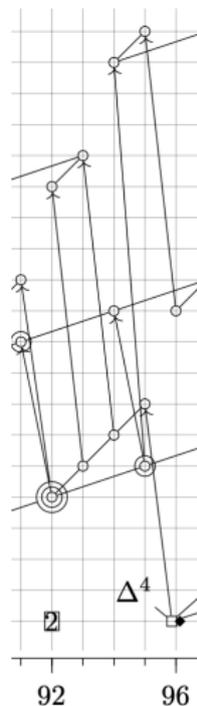
Example: $d_7(\Delta^4)$

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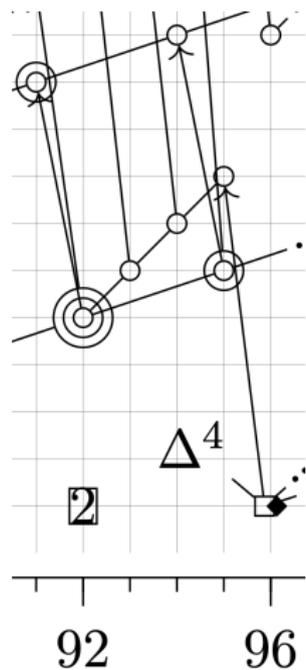
Figure: $d_7(\Delta^4)$ in $\mathrm{DSS}(Tmf)$

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- We need to be late enough to see this extension (Smf/τ^3), but early enough that it hasn't disappeared (Smf/τ^5).

Figure: $d_7(\Delta^4)$ in $\mathrm{DSS}(Tmf)$

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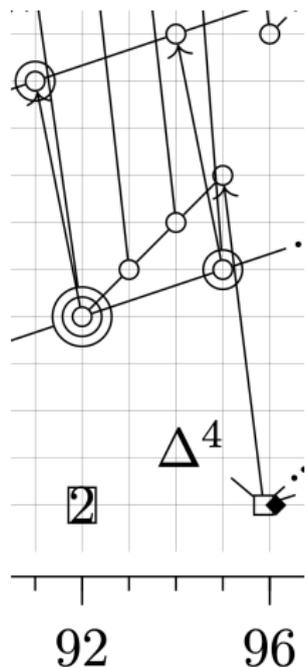
Tool 7: Synthetic and temporary Toda brackets

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- Upshot: this construction makes sense in $\mathrm{Fil}(Sp)$, and the maps of $\mathbb{E}_\infty$-algebras $R \to R/\tau^n$ and $R \to \tau^{-1}R$ preserve Toda brackets.

Example: hidden extension on $2\nu\Delta$.

- $2\nu\Delta$ lifts to Smf/τ^{12}

Figure: E_5-page of $\mathrm{DSS}(Tmf)$

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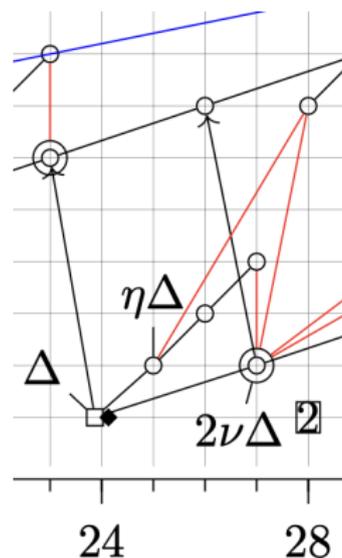- $2\nu\Delta = \langle \bar{\kappa}, \tau^4\nu, 2\nu \rangle$

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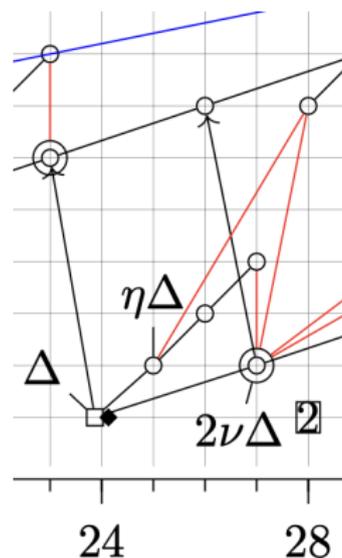
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Since $\pi : R \to R/\tau$ is a map of $\mathbb{E}_\infty$-rings, we have a containment $\pi(\langle x, y, z \rangle) \subset \langle \pi(x), \pi(y), \pi(z) \rangle$.

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- e.g. $\nu \cdot 2\nu = 2\nu \cdot \eta = 0 \in \pi_{*,*}(Smf/\tau)$, and $\epsilon = \langle \nu, 2\nu, \eta \rangle$

Example: 3-fold Moss theorem

Theorem (CDvN)

Let $x, y, z \in \pi_{*,*}R$ with $\tau^r xy = \tau^s yz = 0$ for $r \geq s \geq 0$ minimal.

- *If $s > 0$, let $d_{r+1}(H_0) = \pi(xy)$ and $d_{s+1}(H_1) = \pi(zy)$. Then $\langle x, \tau^r y, z \rangle \subset \pi_{*,*}R$ contains a lift of*

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- $d_5(\Delta) = \nu\bar{\kappa} \implies 2\nu\Delta \in \langle \bar{\kappa}, \tau^4\nu, 2\nu \rangle \subset \pi_{*,*}(Smf/\tau^{12})$, where

$$s = 0, r = 4, H_0 = \Delta, x = \bar{\kappa}, y = \nu, z = 2\nu$$

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- We can get around this by using on-the-nose relations from $\mathrm{ANSS}(\mathbb{S})$ or by working mod a power of τ that forces these relations to hold.

Quick Summary

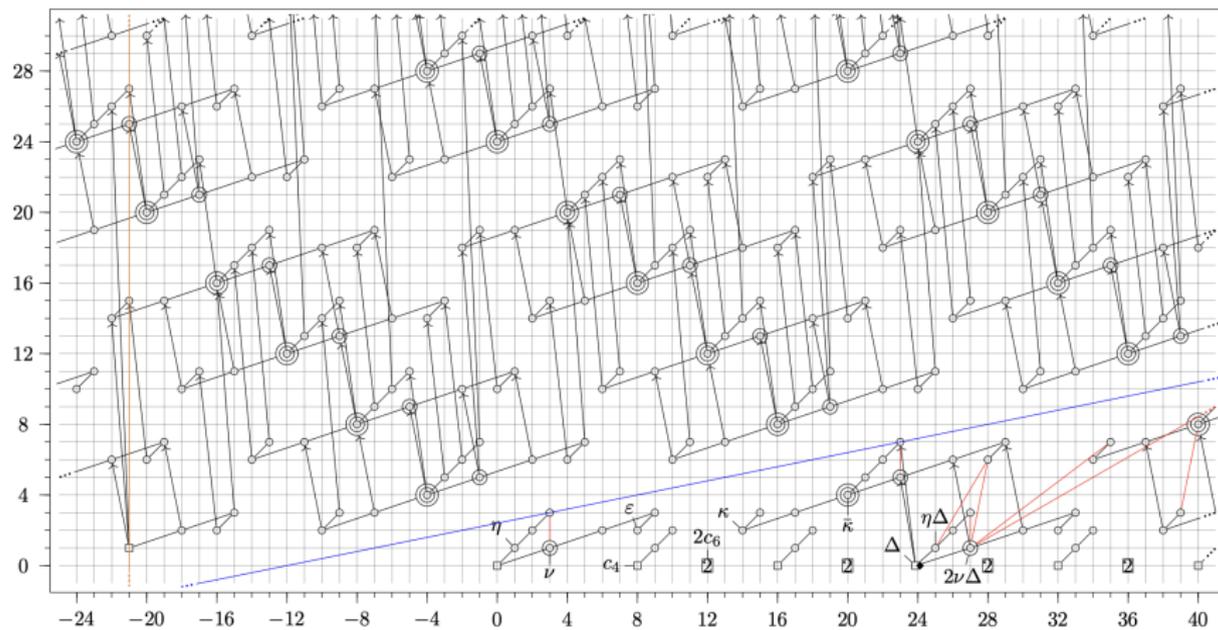Figure: DSS(*Tmf*)

Quick Summary

- Tool 1: The τ-formalism and bigraded homotopy groups
- Tool 2: Designer spectral sequences and maps between them
- Tool 3: Detection methods
- Tool 4: Total differentials
- Tool 5: Working mod τ^n
- Tool 6: The generalized Leibniz rule
- Tool 7: Synthetic and temporary Toda brackets
- Tool 8: Synthetic Moss's theorem

Thank you for your attention