Classifying Spaces and Spectral Sequences

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Introduction

These are a set of expository notes I wrote in preparation for a talk given in the MIT Kan Seminar on December 7, 2016 on Graeme Segal's paper *Classifying Spaces and Spectral Sequences*.

1 Classifying Spaces

Definition 1.1. Let G be a topological group. A principal G-bundle is a space P with a free action of G and an equivariant map $p : P \to B$ for a trivial G-space B such that B has an open covering $\{U_\alpha\}$ with equivariant homeomorphisms $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$ for all α fitting into

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] \arrow[d, "p"] & U_\alpha \times G \arrow[dl, "\mathrm{pr}_1"] \\ U_\alpha & \end{tikzcd}$$

where G is given the standard action on itself. The map $P/G \to B$ is a homeomorphism, so this amounts to saying that P is a locally trivial free G-space with orbit space B.

Definition 1.2. Let G be a topological group. We say a space BG is a classifying space for G if there is a natural isomorphism

$$\{\text{Isomorphism Classes of Principal } G\text{-bundles over } X\} \to [X, BG]$$

for X a CW complex.

Remark 1.3. One can see that if BG exists, it is unique up to weak equivalence by the Yoneda lemma and the fact that every space is weakly equivalent to a CW complex. We will construct two different models of BG, the classical one from Milnor, and one from Segal. The latter will have the advantage that

$$B(G \times G') \cong BG \times BG'$$

Proposition 1.4. If EG is a weakly contractible space with a free action of G such that $EG \to EG/G$ is a principal G-bundle, then $BG := EG/G$ is a classifying space for G as above. $\square$

Remark 1.5. It is not enough to assume that EG is weakly contractible with a free action of G - in fact it's not even enough to assume also that $EG \to EG/G$ is a fiber bundle. Consider the following counterexample: Let G^{ind} be the set G with the indiscrete topology, then every map into G^{ind} is continuous, so G^{ind} is a contractible space with a free G action, and $G^{\text{ind}} \to *$ is a fiber bundle (trivial, in fact). However, $G^{\text{ind}}/G = *$, but $*$ cannot be a classifying space for G unless G is the trivial group. The problem in this example is that the fiber is G^{ind}, not G. $EG \to EG/G$ is a fiber bundle with fiber G, then the statement would be correct.

1.1 Milnor's BG

Definition 1.6. The topological join of two spaces X and Y is defined as

$$X * Y := \frac{X \times Y \times I}{(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1)}$$

So $X * Y$ stretches $X \times Y$ to a cylinder and then collapses 0 endpoint to X and collapses the 1 endpoint to Y. If X is an $(n-1)$ connected space, and Y is $(m-1)$ connected, then $X * Y$ is $n+m$ connected.

Remark 1.7. A point in $X_1 * X_2 * \cdots * X_n$ can be characterized as a symbol $t_1 x_1 \oplus \cdots \oplus t_n x_n$, where $\sum t_i = 1$, $x_i \in X_i$ unless $t_i = 0$, in which case the i-th symbol is omitted. Checking the case $n = 2$ against the above definition illustrates this. One topologizes the set of such symbols by giving it the finest topology such that the maps

$$t_i : X_1 * X_2 * \cdots * X_n \to [0,1] \qquad \text{and} \qquad x_i : t_i^{-1} : (0,1] \to X_i$$

are continuous. □

Definition 1.8. For a topological group G, let $E^n G = G * \cdots * G$, the $(n+1)$-fold self-join of G. $E^n G$ has a free right G-action given by

$$\begin{gathered} E^n G \times G \to E^n G \\ (t_0 g_0 \oplus \cdots \oplus t_n g_n, g) \mapsto t_0(g_0 g) \oplus \cdots \oplus t_n(g_n g) \end{gathered}$$

Lemma 1.9. $p : E^n G \to (E^n G)/G$ is a principal G-bundle.

Proof: Let $U_i = \{p(t_0 g_0 \oplus \cdots \oplus t_n g_n) : t_i \neq 0\}$, then we may define maps

$$\begin{gathered} p^{-1}(U_i) \to U_i \times G \\ t_0 g_0 \oplus \cdots \oplus t_n g_n \mapsto (p(t_0 g_0 \oplus \cdots \oplus t_n g_n), g_i) \end{gathered}$$

$$\begin{gathered} U_i \times G \to p^{-1}(U_i) \\ (p(t_0 g_0 \oplus \cdots \oplus t_n g_n), g) \mapsto t_0(g_0 g_i^{-1} g) \oplus \cdots \oplus t_n(g_n g_i^{-1} g) \end{gathered}$$

One checks that the first is well-defined, and both are continuous, and it is easy to see that they are mutually inverse. The quotient map thus has fiber G, so by the discussion in 1.5, this is a principal G-bundle. □

Proposition 1.10. Let $EG := \mathrm{colim}_n E^n G$, then EG/G is a classifying space for G.

Proof: Since taking iterated joins increases connectivity, EG is weakly contractible. We may characterize EG as the set of symbols as above where only finitely many t_i are nonzero. The same argument then shows that EG is a free G-space, and $EG \to EG/G$ is a fiber bundle. The result therefore follows from 1.4. □

1.2 Segal's BG

Definition 1.11. Let Δ denote the simplex category - its objects are the sets $[n] := \{0, 1, \ldots, n\}$, and its morphisms are nondecreasing functions. A *simplicial object* in a category $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$, and a cosimplicial object is a functor $\Delta \to \mathcal{C}$. (Co)-simplicial objects in $\mathcal{C}$ form a category via natural transformations of functors.

Example 1.12. If $\mathcal{C} = \mathbf{Spaces}$, the sequence of standard topological n-simplices $\Delta[n]$ forms a cosimplicial object in **Spaces** via the usual face and degeneracy maps.

When $\mathcal{C} = \mathbf{Sets}$, we call a simplicial object in $\mathcal{C}$ a simplicial set, and similarly for a simplicial space. Any simplicial set A has a geometric realization

$$|A| := \frac{\coprod_{n \geq 0} \Delta[n] \times A([n])}{(\theta_*(x), a) \sim (x, \theta^*(a))} \in \mathbf{Spaces}$$

where $A([n])$ is given the discrete topology, and θ is a morphism in Δ. We may similarly define the geometric realization of a simplicial space, and these define functors $|-|$: **sSet** $\to$ **Spaces** and $|-|$: **sSpaces** $\to$ **Spaces**, where the notation $s\mathcal{C}$ is the category of simplicial objects in $\mathcal{C}$.

Example 1.13. Let $A(n)$ be the simplicial set sending $[m] \mapsto \mathrm{Hom}_\Delta(-,[n])$. Then $|A(n)| \cong \Delta[n]$ because if $x \in \Delta[k]$ and $\theta \in \mathrm{Hom}_\Delta([k],[n])$, the equivalence relation $(x,\theta) = (x, \theta^*(\mathrm{id}_{[n]})) \sim (\theta_*(x), \mathrm{id}_{[n]})$ implies the map $\Delta[n] \to |A|$ that includes $\Delta[n]$ via $\Delta[n] \times \{\mathrm{id}_{[n]}\}$ on the n-th summand is a homeomorphism.

We may define the product of simplicial sets A, B by taking their levelwise cartesian product in **Sets**, and the map $|A \times B| \to |A| \times |B|$ defined in the obvious way is a bijection. If $|A|$ and $|B|$ are compactly generated spaces, then the map is always a homeomorphism. Since $|-|$ is defined in exactly the same way for simplicial spaces, the corresponding map is again a homeomorphism when we take a simplicial space to mean a simplicial object in the category of compactly generated spaces. For this reason, we now assume **Spaces** to mean the category of compactly generated spaces.

Definition 1.14. Let $[n]$ also denote the category$\{0 \to 1 \to \cdots \to n\}$. The *nerve* of a category $\mathcal{C}$ is the simplicial set $N\mathcal{C}([n]) := \mathrm{Fun}([n], \mathcal{C})$, i.e. the set of functors from $[n]$ to $\mathcal{C}$. Another way to think of this is that $N\mathcal{C}([n])$ is the set of n-simplices formed by commutative diagrams in $\mathcal{C}$ (i.e. 1-simplices are morphisms, 2-simplices are commutative triangles, 3-simplices are commutative tetrahedra, and so on). We let $B\mathcal{C} := |N\mathcal{C}|$ denote the *classifying space* of the category $\mathcal{C}$.

Definition 1.15. We are interested in the case when the nerve of a category is naturally a simplicial space. A *topological category* is a small category (i.e. the objects and morphisms each form a set) $\mathcal{C}$ where the sets $\mathrm{ob}(\mathcal{C})$ and $\mathrm{mor}(\mathcal{C})$ have topologies so that the maps

1. Domain: $\mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$
2. Codomain: $\mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$
3. Identity: $\mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$
4. Composition: $\mathrm{Hom}_\mathcal{C}(A,B) \times \mathrm{Hom}_\mathcal{C}(B,C) \to \mathrm{Hom}_\mathcal{C}(A,C)$, $\forall\ A,B,C \in \mathrm{ob}(\mathcal{C})$

and the sets $\mathrm{Hom}_\mathcal{C}(-,-)$ are given the subspace topology of $\mathrm{mor}(\mathcal{C})$. In this case $N\mathcal{C}$ is a simplicial space.

Proposition 1.16. If $\mathcal{C}$ and $\mathcal{C}'$ are topological categories, and $F_i : \mathcal{C} \to \mathcal{C}'$ for $i = 0,1$ are continuous functors with respect to the topologies on the object and morphism sets, and there is a natural transformation $T : F_0 \to F_1$, then the induced maps $BF_i : B\mathcal{C} \to B\mathcal{C}'$ are homotopic.

Proof: The functor $\mathcal{C} \to N\mathcal{C}$ commutes with products because

$$N(\mathcal{C} \times \mathcal{C}')[n] = \mathrm{Fun}([n], \mathcal{C} \times \mathcal{C}') \cong \mathrm{Fun}([n], \mathcal{C}) \times \mathrm{Fun}([n], \mathcal{C}')$$

hence $\mathcal{C} \mapsto B\mathcal{C}$ commutes with products as long as all spaces are compactly generated. A natural transformation $T : F_0 \to F_1$ is the same data as a functor $\mathcal{C} \times [1] \to \mathcal{C}'$ where $[1]$ is the category as in 2.1, and F_i is the functor $\mathcal{C} \times \{i\} \hookrightarrow \mathcal{C} \times [1] \to \mathcal{C}'$. Hence there is map $BT : B(\mathcal{C} \times [1]) \to B\mathcal{C}'$, but $B(\mathcal{C} \times [1]) \cong B\mathcal{C} \times B[1]$, and $N([1])$ is the simplicial set of 1.13 since a functor from $[n] \to [1]$ is the same as a morphism in Δ from the ordered sets of the same names. Hence $B[1]$ is the topological 1 simplex, which we may identify with $[0,1] \subset \mathbb{R}$. BT therefore defines a homotopy between BF_0 and BF_1. $\square$

Remark 1.17. It follows from 1.16 that if $\mathcal{C}$ has an initial or terminal object, then $B\mathcal{C}$ is a contractible space because there is a natural transformation from the identity functor to the constant functor at the terminal object, and the other way around for an initial object.

If G is a topological group, we let G also denote the topological category with $\mathrm{ob}(G) = *$ and $\mathrm{mor}(G) = G$, and we show that the space BG as we have defined it is a classifying space for G in many cases. We note that $NG[k] \cong G^k$, and the various face and degeneracy maps are given by projections onto factors, multiplication in G, and the inclusions $G \to G \times G$ on the left and right via the identity of G.

Let $\overline{G}$ be the category with $\mathrm{ob}(G) = G$ and $\mathrm{mor}(G) = G \times G$, so that there is a unique morphism $g \to h$ which we may think of as multiplication on the right by $g^{-1}h$; indeed there is a functor $p : \overline{G} \to G$ sending the morphism $g \to h$ to $g^{-1}h$. Now, $N\overline{G}[k] \cong G^{k+1}$, and the face and degeneracy maps are given by projections onto factors and diagonal maps. G acts on $N\overline{G}$ by acting levelwise via the diagonal action $g \cdot (g_1, \ldots, g_{k+1}) = (gg_1, \ldots, gg_{k+1})$, and the face and degeneracy maps are equivariant with respect to these actions, which is to say $N\overline{G}$ is a simplicial object in the category of G-spaces. It thus follows that taking the levelwise quotient by the action of G gives a simplicial space $N\overline{G}/G$. In fact the map $Np : N\overline{G} \to NG$ factors through $N\overline{G}/G$ because, on the k-th level

$$\begin{aligned} Np(g(g_1, \ldots, g_k)) &= Np(gg_1, \ldots, gg_{k+1}) \\ &= (g_1^{-1}g^{-1}gg_2, \ldots, g_k^{-1}g^{-1}gg_{k+1}) \\ &= (g_1^{-1}g_2, \ldots, g_k^{-1}g_{k+1}) \\ &= Np(g_1, \ldots, g_{k+1}) \end{aligned}$$

and $N\overline{G}/G \to NG$ is an isomorphism as it has the inverse given levelwise by

$$(g_1, \ldots, g_k) \mapsto ((g_1 \cdots g_k)^{-1}, (g_2 \cdots g_k)^{-1}, \ldots, (g_{k-1}g_k)^{-1}, e)$$

It follows that $B\overline{G}/G \to BG$ is an isomorphism since the quotients by the action of G levelwise and the quotient defining geometric realization can be taken in either order. Taking the quotient by the group first gives $|NG/G| \cong BG$, and taking it second gives $B\overline{G}/G$. G acts freely on $N\overline{G}$ since the diagonal action is free, hence it acts freely on $B\overline{G}$. $\overline{G}$ has an initial object given by the identity element of G, hence $B\overline{G}$ is contractible by 1.16. BG would thus be a classifying space for G if we knew that $B\overline{G} \to BG$ were a fiber bundle with fiber G. It turns out that this is the case whenever G is an absolute neighborhood retract, for instance if G is a locally finite CW complex, or a topological manifold.

Milnor's construction can also be phrased in terms of the nerve. In particular, if $\mathcal{C}$ is a topological category, and $\mathbb{N}$ is the category of natural numbers considered as an ordered set, then let $\mathcal{C}_{\mathbb{N}}$ be the subcategory of $\mathbb{N} \times \mathcal{C}$ given by deleting non-identity morphisms of the form $(n, c) \to (n, c')$. Then $B\overline{G}_{\mathbb{N}} \cong G * G * \cdots$ and $BG_{\mathbb{N}} \cong (G * G * \cdots)/G$. $B\overline{G}_{\mathbb{N}}$ is contractible since $\overline{G}_{\mathbb{N}}$ has the initial object $(0, e)$. This construction of the classifying space fails to satisfy $B(G \times G') \cong BG \times BG'$ since one easily checks that $(G \times G')_{\mathbb{N}} \cong (G_{\mathbb{N}}) \times_{\mathbb{N}} (G'_{\mathbb{N}})$ and therefore $B(G \times G') \cong BG \times_{B\mathbb{N}} BG'$ and $B\mathbb{N}$ is the infinite simplex. Note, by uniquess of BG (1.3), one has a weak equivalence $B(G \times G') \cong BG \times BG'$ in either case, but only in Segal's BG do we always have a homeomorphism, since $B = |N(-)|$ commutes with products.

2 Spectral Sequences

Definition 2.1. A (bigraded, homological) Spectral Sequence is a sequence of bigraded abelian groups $E^r_{p,q}$ for $r \geq 1$ with differentials $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $E^{r+1} = H(E^r, d_r)$. If C_* is a graded abelian group with a filtration

$$0 \subset F_1C_* \subset F_2C_* \subset \cdots \subset C_*$$

with $\bigcup F_iC_* = C_*$, (E^r, d_r) is said to converge to C_* if for all p, q, there exists $r(p,q)$ such that $E^r_{p,q} = E^{r(p,q)}_{p,q}$ for $r \geq r(p,q)$, and $E^\infty_{p,q} := E^{r(p,q)}_{p,q} \cong F_pC_{p+q}/F_{p-1}C_{p+q}$. When (E^r, d_r) converges to C_*, one often uses the notation

$$E^2_{p,q} \implies C_*$$

2.1 Exact Couples and Filtered Complexes

Definition 2.2. An exact couple is a diagram of abelian groups

$$\begin{tikzcd} D^1 \arrow[rr, "i"] & & D^1 \arrow[dl, "j"] \\ & E^1 \arrow[ul, "k"] & \end{tikzcd}$$

that is exact at each node.

Proposition 2.3. For an exact couple, if we set $D^2 = \mathrm{im}(i)$ and $E^2 = \mathbf{ker}(j \circ k)/\mathrm{im}(j \circ k)$, then

$$\begin{tikzcd} D^2 \arrow[rr, "i|_{\mathrm{im}(i)}"] & & D^2 \arrow[dl, "{[j \circ i^{-1}]}"] \\ & E^2 \arrow[ul, "k"] & \end{tikzcd}$$

is an exact couple. An exact couple thus determines a spectral sequence by iterating this process and taking $j \circ k$ to be the differential at each level.

Proof: By $[j \circ i^{-1}]$ we mean taking a preimage under i, applying j and then taking homology. It is easy to check that the maps are well defined and that the diagram is exact at each node. ☐

Definition 2.4. A filtered chain complex C_* is a filtered graded abelian group as above such that restricting the differential to each piece of the filtration gives a chain complex. In other words C^* is a filtered object in the category of chain complexes.

Proposition 2.5. A filtered chain complex determines an exact couple and thus a spectral sequence.

Proof: Set $D^1 = \bigoplus_{p,q} H_{p+q}(F_p C_*)$, $E^1 = \bigoplus_{p,q} H_{p+q}(F_p C_*/F_{p-1}C_*)$, then since a short exact sequence of chain complexes determines a long exact sequence in homology, these form an exact couple. ☐

Definition 2.6. A bounded spectral sequence (E^r, d_r) is one such that for all n, r, the set

$$\{E^r_{k,n-k} \text{ is nonvanishing}\}$$

is finite. All bounded spectral sequences converge because this property implies that for any p, q, for $r >> 0$ the differentials entering and exiting $E^r_{p,q}$ are zero.

Proposition 2.7. If the spectral sequence of a filtered complex is bounded, it converges to $H_*(C_*)$, where the filtration on $H_*(C_*)$ is given by

$$\overline{F}_p H_*(C_*) = \mathrm{im}(H_*(F_p C_*) \to H_*(C_*))$$

Proof: A bit of diagram chasing shows that

$$E^r_{p,q} = \frac{\{c \in F_p C_{p+q} : \partial(c) \in F_{p-r} C_{p+q-1}\}/F_{p-1}C_{p+q}}{\partial(F_{p+r-1}C_{p+q+1})}$$

in the case $r = 1$, this is straight from the definition. If the spectral sequence is bounded, then we may take $r \to \infty$ and we find

$$E^\infty_{p,q} = \frac{\{c \in F_p C^{p+q} : \partial(c) = 0\}/F_{p-1}C_{p+q}}{\partial(C_{p+q+1})} = \frac{\mathrm{im}(H_{p+q}(F_p C_*) \to H_{p+q}(C_*))}{\mathrm{im}(H_{p+q}(F_{p-1}C_*) \to H_{p+q}(C_*))}$$

☐

2.2 Examples

We go through a few examples of spectral sequences, each one being a generalization of the previous, and we arrive at the spectral sequence given by Segal.

Example 2.8. Let X be a finite dimensional CW complex with a skeletal filtration

$$\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

i.e. X_i is the i-skeleton of X. Then we filter the singular chain complex $C_*(X)$ by setting $F_pC_*(X) = C_*(X_p)$. Then we get an exact couple

$$\begin{tikzcd} \bigoplus_{p,q} H_{p+q}(X_p) \arrow[rr] & & \bigoplus_{p,q} H_{p+q}(X_p) \arrow[dl] \\ & \bigoplus_{p,q} H_{p+q}(X_p, X_{p-1}) \arrow[ul] & \end{tikzcd}$$

and we recall that

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}) = \begin{cases} C^{\text{cell}}_p(X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

since X_p/X_{p-1} is a wedge of spheres - one for each p-cell of X. Then since the filtration is bounded, the spectral sequence is bounded, so it converges to $H^*(X)$. It is easy to check that the boundary map in the spectral sequence is just the boundary map in the long exact sequence of a triple, which is by definition the cellular boundary map. We thus have that

$$E^\infty_{p,q} = \begin{cases} H^{\text{cell}}_p(X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

The fact that this spectral sequences converges to the graded object $H_*(X)$ then says that in the filtration

$$0 \subset \operatorname{im}(H_p(X_0) \to H_p(X)) \subset \cdots \subset \operatorname{im}(H_p(X_n) \to H_p(X)) = H_p(X)$$

if we set $F_k = \operatorname{im}(H_p(X_n) \to H_p(X))$, then $E^\infty_{k,p-k} \cong F_k/F_{k-1}$. Then since $E^\infty_{k,p-k}$ is nonvanishing only if $k = p$, the above filtration collapses to

$$0 = \operatorname{im}(H_p(X_{p-1}) \to H_p(X)) \subset \operatorname{im}(H_p(X_p) \to H_p(X)) = H_p(X)$$

and we thus have $H^{\text{cell}}_p(X) \cong H_p(X)$.

Definition 2.9. A generalized homology theory k is a sequence of functors k_n from the category of pairs of spaces to the category of abelian groups, together with a natural transformation $\partial : k_i(X, A) \to k_{i-1}(A, \emptyset)$ for each i satisfying the following axioms:

1. Homotopy: Homotopic maps induce the same maps in homology
2. Excision: If (X, A) is a pair and U is a subset of X such that the closure of U is contained in the interior of A, then the inclusion map $i : (X - U, A - U) \to (X, A) \to (X, A)$ induces an isomorphism in homology.
3. Additivity: If $X = \coprod_\alpha X_\alpha$, the disjoint union of a family of topological spaces X_α, then $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$
4. Long exact sequences: each pair (X, A) induces a long exact sequence in homology, via the inclusions $i : A \to X$ and $j : X \to (X, A)$, and ∂.

Example 2.10. (The Atiyah-Hirzebruch Spectral Sequence) We generalize the previous spectral sequence to that of a generalized homology theory. For a generalized theory k_*, we no longer have the description of k_* as the homology of some chain complex, so we need to modify the above arguments. In the same situation as above we use the long exact sequence axiom to obtain an exact couple

$$\begin{tikzcd}
\bigoplus_{p,q} k_{p+q}(X_p) \arrow[rr] & & \bigoplus_{p,q} k_{p+q}(X_p) \arrow[dl] \\
& \bigoplus_{p,q} k_{p+q}(X_p, X_{p-1}) \arrow[ul] &
\end{tikzcd}$$

Since X is once again assumed to be finite dimensional, the spectral sequence is bounded and it converges. It converges to $k_*(X)$ which one can see by modifying 2.7 a bit via some diagram chasing. We can still identify thq quotient X_p/X_{p-1} as a wedge of spheres, so we have

$$\begin{aligned}
k_{p+q}(X_p, X_{p-1}) &\cong \widetilde{k_{p+q}}(X_p/X_{p-1}) \\
&\cong \widetilde{k_{p+q}}(\bigvee_\alpha S^p) \quad \text{where } \alpha \text{ runs over the } p-\text{cells of } X \\
&\cong \bigoplus_\alpha k_q(*) \\
&\cong C_p^{\text{cell}}(X; k_q(*))
\end{aligned}$$

where we define cellular homology with coefficients in G in the same way as with singular homology - by tensoring with the group ring of G. Just as above, the boundary map in the spectral sequence corresponds under these isomorphisms to the boundary map in the long exact sequence of a triple and thus cellular boundary map, then since we have shown cellular homology to be the same as ordinary homology, we have

$$H_p(X; k_q(*)) \implies k_{p+q}(X)$$

Using the same methods in cohomology, we have

$$H^p(X; k^q(*)) \implies k^{p+q}(X)$$

Example 2.11. (Segal's Spectral Sequence) We want to apply the same reasoning to a simplicial space A by taking a filtration of its geometric realization. Indeed $|A|$ has a natural filtration: namely let $|A|_p$ be the image of the map

$$\Delta[p] \times A_p \to \left(\coprod_{n \geq 0} \Delta[n] \times A_n \right) / \sim = |A|$$

onto the p-th summand. Then as before (this time using cohomology) we get a spectral sequence beginning with $E_1^{p,q} = k^{p+q}(|A|_p, |A|_{p-1})$. This is no longer a skeletal filtration but we can still identify the quotients $|A|_p/|A|_{p-1}$, and Segal's key observation is that there is a relative homeomorphism $(|A|_p, |A|_{p-1}) \cong (\Sigma^p A_p, \Sigma^p A_p^d)$ where A_p^d is the degenerate part of A_p (i.e. the union of the images of all the maps $A_k \to A_p$ with $k < p$). With a little more work, Segal identifies $E_1^{p,q}$ with $k^q(A_p)$. To finish the analogy with cellular homology, we need to identify the differential in the spectral sequence with a differential in another cochain complex. To this end, applying k^q to the simplicial space A one has a cochain complex with differential $k^q(A_p) \to k^q(A_{p+1})$ given by

$$k^q(A_p) \xrightarrow{\theta} \prod_p k^q(A_{p+1}) \xrightarrow{\Sigma} k^q(A_{p+1})$$

where θ is the product of the maps induced by the $p+2$ injections $[p] \to [p+1]$, and Σ is summation with alternating signs, and one finds that this differential corresponds to the differential of the spectral sequence. We therefore have

$$H^p(k^q(A)) \implies k^{p+q}(|A|)$$

Applying this to the simplicial space NG, we have a spectral sequence computing the k^*-cohomology of BG that begins with the cohomology of the simplicial cochain complex

$$k^*(*) \to k^*(G) \to k^*(G \times G) \to \cdots$$

and this is often called the bar construction. We thus have a tool, for instance, to compute the coefficient ring of equivariant k-theory for any group G.