Computations in Stable Motivic Homotopy

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The goal of this seminar is to understand some fundamental computational tools in stable motivic homotopy theory. We will begin by constructing the unstable and stable motivic homotopy categories. For our first computation, we will explore Morel's computation of π_0 of the motivic sphere spectrum as the Grothendieck-Witt ring. We then move to Adams spectral sequences, exploring \mathbb{C} and \mathbb{R} -motivic stable homotopy through the relevant dual Steenrod algebra computations and the recent deformation perspectives on these homotopy theories. We finish by exploring the connections between \mathbb{R} -motivic stable homotopy and C_2 -equivariant stable homotopy, as well as rephrasing some of the deformation results in terms of Pstragowski's synthetic spectra.

Talk 1. Unstable Motivic Homotopy, Part 1

Motivation: classifying algebraic vector bundles, Weil conjectures. Construction of the \mathbb{A}^1 -homotopy category and basic examples: sheaf of \mathbb{A}^1 -connected components, bigraded spheres.

Sources: [Antieau-Elmanto] sections 2-4, [Dugger] part 1

Talk 2. Unstable Motivic Homotopy, Part 2

Survey of examples in unstable motivic homotopy. Classifying spaces, algebraic K-theory, Purity, classification of vector bundles.

Sources: [Antieau-Elmanto] sections 4-9

Talk 3. Stable Motivic Homotopy

Construct the motivic stable homotopy category, go thru examples: HZ, MGL, KGL, the slice filtration.

Sources: [Dundas] Part 3

Talk 4. Stable π_0 is the Grothendieck-Witt Ring

Prove that π_0 of the motivic sphere spectrum is the Grothendieck-Witt Ring.

Sources: [Morel], [Bachmann-Østvaer]

Talk 5. C-Motivic Stable Homotopy

Calculate the \mathbb{C} -motivic homology of a point and dual Steenrod algebra. Discuss the characterization of \mathbb{C} -motivic homotopy as a deformation of classical stable homotopy with parameter τ .

Sources: [Voe1], [Voe2], [GWX], [GIKR]

Talk 6. \mathbb{R} -Motivic Stable Homotopy

Calculate the \mathbb{R} -motivic homology of a point and dual Steenrod algebra. Discuss the characterization of \mathbb{R} -motivic homotopy as a deformation of classical stable homotopy with parameter ρ .

Sources: [Voe1], [Voe2], [Behrens-Shah]

Talk 7. C₂-Equivariant Dual Steenrod Algebra

Go through computation of the C₂-equivariant homology of a point and dual Steenrod algebra. Discuss a_{σ} and the exact sequence relating it to $RO(C_2)$ -graded Mackey functor structure.

Sources: [Greenlees], [Wilson], [Hu-Kriz]

Talk 8. \mathbb{R} -Motivic Homotopy and C₂-equivariant Homotopy

Construct the Betti realization functor from the \mathbb{R} -motivic stable homotopy category to the C₂-equivariant stable homotopy category. Explain how this realizes the corresponding maps on dual Steenrod algebras and Adams spectral sequences and carry out some example computations

Sources: [Behrens-Shah], [Heller-Ormsby], [Burklund-Hahn-Senger]

Talk 9. Bredon-Landweber region in C₂-equivariant homotopy Bruner-Greenlees reformulation of Mahowald invariant. Computations in Guillou-Isaksen

Sources: [Bruner-Greenlees], [Guillou-Isaksen]

Talk 10. Synthetic Spectra

Revisit the deformation discussions from Talks 5 and 6 in terms of Pstragowski's synthetic spectra. BP-synthetic spectra = \mathbb{C} -motivic spectra. HF₂-synthetic spectra.

Sources: [Pstragowski], [Burklund]

References

[Antieau-Elmanto] A Primer for Unstable Motivic Homotopy, Benjamin Antieau and Elden Elmanto

[Behrens-Shah] C_2 -equivariant Stable Homotopy from Real Motivic Stable Homotopy, Mark Behrens, Jay Shah

[Bruner-Greenlees] The Bredon-Löffler Conjecture, Bob Bruner, J.P.C. Greenlees

[Burklund] An Extension in the Adams Spectral Sequence in Dimension 54, Robert Burklund

[Burklund-Hahn-Senger] Galois Reconstruction of Artin-Tate \mathbb{R} -Motivic Spectra, Robert Burklund, Jeremy Hahn, Andy Senger

[Dugger] Navigating the Motivic World, Daniel Dugger

[Dundas] *Motivic Homotopy Theory*, Bjørn Dundas, Marc Levine, Paul Østvaer, Oliver Röndigs, Vladimir Voevodsky

[GWX] The Special Fiber of the Motivic Deformation of the Stable Homotopy Category is Algebraic, Bogdan Gheorghe, Guozhen Wang, Zhouli Xu

 $[{\rm GIKR}]$ $\mathbb C$ -Motivic Modular Forms, Bogdan Gheorghe, Dan Isaksen, Achim Krause, Nickolas Ricka

[Greenlees] Four Approaches to Cohomology Theories with Reality, J.P.C. Greenlees

 $[\mbox{Guillou-Isaksen}] \ The \ Bredon-Landweber \ Region \ in \ C_2-equivariant \ Stable \ Homotopy \ Groups, \ Bert \ Guillou, \ Dan \ Isaksen$

[Heller-Ormsby] Galois Equivariance and Stable Motivic Homotopy Theory, Jeremiah Heller, Kyle Orsmby

[Hu-Kriz] Real-Oriented Homotopy Theory and an Analogue of the Adams-Novikov Spectral Sequence, Po Hu, Igor Kriz

[Morel] On the Motivic π_0 of the Sphere Spectrum, Fabien Morel

[Pstragowski] Synthetic Spectra and the Cellular Motivic Category, Piotr Pstragowski

- [Voe1] Reduced Power Operations in Motivic Cohomology, Vladimir Voevodsky
- [Voe2] Motivic Cohomology with Z/2 Coefficients, Vladimir Voevodsky]

[Wilson] Power Operations for $H\mathbb{F}_2$ and a Cellular Construction of $BP_{\mathbb{R}}$, Dylan Wilson