Higher Real K-Theories and the Slice Spectral Sequence

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The goal of this seminar is to explore recent advances in the study of the Hopkins-Miller Higher real K-Theory spectra, particularly at the prime 2. A "Higher real K-theory" is the homotopy fixed point spectrum of a Morava E-theory under the action of a finite subgroup of the Morava stabilizer group. For example, 2-complete complex K-theory KU is a height 1 Morava E-theory, and its homotopy fixed points under the C_2 action via formal inversion is 2-complete real K-theory KO. As an abuse of notation, the choice of Morava E-theory and subgroup is often suppressed, and we simply refer to a Higher real K-theory as an "EO_n".

As the height n grows, EO_n's detect more information about the stable homotopy category, but the computations become correspondingly difficult. Recent advances (e.g. [HS]) have shown that, at the prime 2, these computations can be controlled by the more geometric Fujii-Landweber Real bordism theory $MU_{\mathbb{R}}$ and its norms $MU^{((G))}$. These have been studied extensively via the slice filtration in [HHR]. We will introduce the slice filtration and review the results of [HHR] on the slices of Real bordism theories, discuss connections to EO_n's, and highlight some of the recent computations these connections have allowed.

Below are brief summaries of each talk with suggestions on what material to cover.

Talk 1. ***Introduction and Odd Primary Kervaire Invariant***

Ravenel's odd primary Kervaire invariant argument as an antecedent to HHR.

References: [Ravenel]

Talk 2. ***Real Orientations***

Discuss $RO(G)$-gradings and representation spheres. Define Real orientations, $MU_{\mathbb{R}}$, $BP_{\mathbb{R}}$, their norms. Establish analogies to chromatic: e.g. $E \wedge MU_{\mathbb{R}} = E[b_1, b_2, \ldots]$ for E Real oriented. and construction of $E_{\mathbb{R}}(n)$'s and $K_{\mathbb{R}}(n)$'s. Atiyah's $K_{\mathbb{R}}$ is Real oriented. Discuss isotropy separation sequence and Tate squares. Discuss geometric fixed points functors, do example induction argument showing $\{\Phi^H(-)\}$ are jointly conservative. Show $\Phi^{C_2}(MU_{\mathbb{R}}) = MO$.

References: [HHRBook] 12.2, [HK] Section 2.

Talk 3. ***Slice Filtration and Slice Spectral Sequence***

Define the regular slice filtration as in [HHRBook], give [HY] geometric connectivity definition, and give some of the proof that these are the same. As an example, show that C_2 slices are determined by $\underline{\pi}_{*\rho}(-)$ and $\underline{\pi}_{*\rho-1}(-)$ directly using the $C_{2+} \to S^0 \to S^\sigma$ cofibration. Discuss Slice tower and $\mathbb{Z}$ and $RO(G)$ graded Slice SS's. Run the $RO(C_2)$ Slice SS for $MU_{\mathbb{R}}$ as an example, taking the slices of $MU_{\mathbb{R}}$ as well as the computation of $\pi_\star H\underline{Z}$ as given.

References: [HHRBook] Sections 11, 13.1, [Hill], [LSWX] Section 3, [Wilson],

Talk 4. ***HHR Slice Theorem***

Discuss the HHR algebra generators for $\pi_*^u MU^{((G))}$. Discuss inductive approach to slice theorem and reduction theorem. Do explicit example with $BP^{((C_4))}$, write out its slices, deduce the SS collapses

in the $*\rho_4$ and $*\rho_4 - 1$ stems, deduce computation of $\pi_{*\rho_4} BP^{((C_4))}$. Slice Differentials theorem.

References: [HHRBook] Sections 12, 13.3

Talk 5. *Slice E_2 Pages*
$\pi_\star H\underline{\mathbb{Z}}$. Homological algebra in $\underline{\mathbb{Z}}$-modules, various computations with the Gold elements. Gap theorem.

References: [Zeng], [Greenlees], [HHRBook] Sections 9.9, 13.2

Talk 6. *Hahn-Shi I: Theory*
Talk about E_n's and EO_n's in general and Goerss-Hopkins-Miller. Discuss some basic examples of actions by finite subgroups of the Morava Stabilizer group via roots of unity. Discuss $EO_1 = \hat{KO}_2$ - the Real orientation of Atiyah's $K_{\mathbb{R}}$ is an antecedent of the [HS] result. Show the E_n's are real oriented following [HS].

References: [HS]

Talk 7. *Hahn-Shi II: Computations*
Discuss the HFPSS, with just $\mathbb{Z}$-grading first, KO example. Discuss $RO(G)$ HFPSS - warmup: compute $\pi_\star F(EC_{2+}, H\mathbb{Z})$, $\pi_\star E_{\mathbb{R}}(2)$. Discuss the map from the Slice SS to the HFPSS, how this gives us HFPSS differentials that generalize the KO one. Go thru [HS] computation in Section 6.

References: [HS], [HM]

Talk 8. *Hurewicz Images*
Discuss various equivariant dual Steenrod algebras. Go thru LSWX Hurewicz Images paper.

References: [HK], [LSWX]

Talk 9. *Models of Lubin-Tate Spectra*
Go thru [BHSZ].

References: [BHSZ]

Talk 10. TBD

References

[BHSZ] *Models of Lubin-Tate spectra via Real bordism theory*, Beaudry, Hill, Shi, Zeng

[Greenlees] *Four Approaches to Cohomology Theories with Reality*, Greenlees

[HHR] *On the Non-Existence of Elements of Kervaire Invariant One*, Hill, Hopkins, Ravenel

[HHRBook] *Equivariant Stable Homotopy Theory and the Kervaire Invariant Problem*, Hill, Hopkins, Ravenel

[Hill] *The Equivariant Slice Filtration: a Primer*, Hill

[HK] *Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence*, Hu, Kriz

[HM] *The C_2-spectrum $Tmf_1(3)$ and its invertible modules*, Hill, Meier

[HS] *Real Orientations of Lubin-Tate Spectra*, Hahn, Shi

[LSWX] *Hurewicz Images of Real Bordism Theory and Real Johnson-Wilson Theory*, Li, Shi, Wang, Xu

[Nave] *The cohomology of finite subgroups of Morava stabilizer groups and Smith-Toda complexes*, Nave

[Ravenel] *The non-existence of odd primary Arf invariant elements in stable homotopy*, Ravenel

[Wilson] *On Categories of Slices*, Wilson

[Zeng] *Equivariant Eilenberg-Maclane Spectra in Cyclic p-Groups*, Zeng